

## 2.1 Matrix Operations

If  $A$  is an  $m \times n$  matrix, then the scalar entry in row  $i$  and column  $j$  is denoted  $a_{ij}$  and is called the  $(i,j)$ -entry of  $A$ .

$$A = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

The diagonal entries in  $A$  are  $a_{11}, a_{22}, a_{33}, \dots$ , and they form the main diagonal of  $A$ .

A diagonal matrix is a square  $n \times n$  matrix whose nondiagonal entries are zero.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \text{a diagonal matrix}$$

An  $m \times n$  matrix whose entries are all zero is a zero matrix and is written as  $0$ .

### Sums and Scalar Multiples

Two matrices are equal if they have the same size (i.e., the same number of rows and columns) and their corresponding columns are equal.

If  $A$  and  $B$  are  $m \times n$  matrices, then the sum  $A+B$  is the  $m \times n$  matrix whose columns are the sums of corresponding columns of  $A$  and  $B$ .

Example 1. If  $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$ , then

$$A+B = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}.$$

If  $r$  is a scalar and  $A$  is a matrix, then the scalar multiple  $rA$  is the matrix whose columns are  $r$  times the corresponding columns of  $A$ . We write  $-A$  for  $(-1)A$ .

Example 2. If  $B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$ , then  $2B = \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}$ .

Theorem 1. Let  $A$ ,  $B$ , and  $C$  be matrices of the same size, and let  $r$  and  $s$  be scalars. Then

- a.  $\underline{A+B=B+A}$
- b.  $\underline{(A+B)+C=A+(B+C)}$
- c.  $\underline{A+0=A}$
- d.  $\underline{r(A+B)=rA+rB}$
- e.  $\underline{(r+s)A=rA+sA}$
- f.  $\underline{r(sA)=(rs)A}$ .

## Matrix Multiplication

Definition. If  $A$  is an  $m \times n$  matrix, and if  $B$  is an  $n \times p$  matrix with columns  $\bar{b}_1, \dots, \bar{b}_p$ , then the product  $AB$  is the  $m \times p$  matrix whose columns are

$A\bar{b}_1, \dots, A\bar{b}_p$ :

$$AB = A[\bar{b}_1 \dots \bar{b}_p] = [A\bar{b}_1 \dots A\bar{b}_p].$$

Example. Compute  $AB$ , where

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}.$$

Solution. Let  $\bar{b}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ ,  $\bar{b}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ ,  $\bar{b}_3 = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ . Then

$$A\bar{b}_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}, \quad A\bar{b}_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \end{bmatrix},$$

$$A\bar{b}_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}.$$

$$\text{So, } AB = [A\bar{b}_1 \ A\bar{b}_2 \ A\bar{b}_3] = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}.$$

## Row-Column Rule for Computing $AB$ .

If the product  $AB$  of matrices  $A$  and  $B$  is defined, then the  $(i,j)$ -entry in  $AB$ , denoted  $(AB)_{ij}$ , is

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}.$$

Example 5. Use the row-column rule to compute the product  $AB$ , where

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}.$$

Solution.

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 2 \cdot 4 + 3 \cdot 1 & 2 \cdot 3 + 3 \cdot (-2) & 2 \cdot 6 + 3 \cdot 3 \\ 1 \cdot 4 + (-5) \cdot 1 & 1 \cdot 3 + (-5) \cdot (-2) & 1 \cdot 6 + (-5) \cdot 3 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 0 & 21 \\ -1 & -7 & -9 \end{bmatrix}. \end{aligned}$$

### Properties of Matrix Multiplication

Theorem 2. Let  $A$  be an  $m \times n$  matrix, and let  $B$  and  $C$  have sizes for which the indicated sums and products are defined. Then

- a.  $A(BC) = (AB)C$
- b.  $A(B+C) = AB+AC$
- c.  $(B+C)A = BA+CA$
- d.  $r(AB) = (rA)B = A(rB)$
- e.  $I_m A = A = A I_n$ .

### Warnings :

1. In general,  $AB \neq BA$ .
  2. If  $AB=AC$ , then it is not true in general that  $B=C$ .
  3. If  $AB=0$ , then you cannot conclude that one of the matrices A and B is 0.
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Example. Let  $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$ . Then

$$AB = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix} \neq \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix} = BA.$$


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### Powers of a Matrix

If A is an  $n \times n$  matrix and  $k$  is a positive integer, then we let  $A^k = A \cdot \dots \cdot A$   $k$  times.

Also,  $A^0 = I$ .

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### The Transpose of a Matrix

Definition. Given an  $m \times n$  matrix A, the transpose of A, denoted  $A^T$ , is the  $n \times m$  matrix whose columns are formed from the corresponding rows of A.

Example. If  $B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}$ , then  $B^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}$ .

Theorem 3. Let A and B denote matrices whose size are appropriate for the following sums and products.

- a.  $(A^T)^T = A$
- b.  $(A + B)^T = A^T + B^T$
- c.  $(rA)^T = rA^T$
- d.  $(AB)^T = B^T A^T$ .