1.9. The Matrix of a Linear Transformation

Theorem. Let $T: R^n \to R^m$ be a linear transformation. Then there exists a unique matrix $\underline{A}$ such that

$$T(\bar{x}) = A\bar{x} \quad \text{for all } \bar{x} \text{ in } R^n.$$

Moreover, if $\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \bar{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ then

$$\underline{A} = [\,T(\bar{e}_1) \;\; T(\bar{e}_2) \;\; \dots \;\; T(\bar{e}_n)\,].$$

Proof. If $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ \vdots \\ 0 \end{bmatrix} + \dots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ x_n \end{bmatrix} =$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = x_1\bar{e}_1 + x_2\bar{e}_2 + \dots + x_n\bar{e}_n,$$

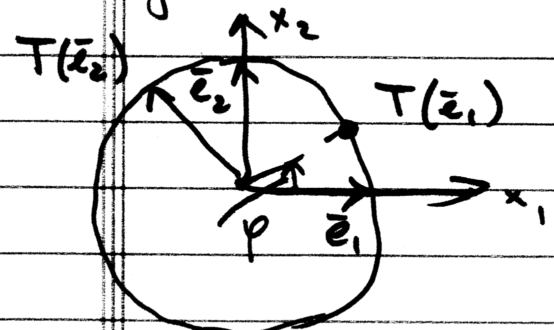
then $T(\bar{x}) = T(x_1\bar{e}_1 + x_2\bar{e}_2 + \dots + x_n\bar{e}_n) =$

$$= x_1T(\bar{e}_1) + x_2T(\bar{e}_2) + \dots + x_nT(\bar{e}_n) =$$

$$= [\,T(\bar{e}_1)\; T(\bar{e}_2) \dots T(\bar{e}_n)\,] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\bar{x}. \quad //$$

The matrix $A = [T(\bar{e}_1) \dots T(e_n)]$ is called the *standard matrix* for the linear transformation T.

Ex. Let $T: R^2 \to R^2$ be the transformation that rotates each point in R^2 about the origin through an angle φ, with counterclockwise rotation.



$$T(\bar{e}_1) = (\cos\varphi, \sin\varphi)$$
$$T(\bar{e}_2) = (-\sin\varphi, \cos\varphi)$$

Finally, $A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$.

$$T(\bar{x}) = A\bar{x}$$

Definition. A mapping $T: R^n \to R^m$ is said to be onto R^m if each $\bar{b}$ in R^m is the image of at least one $\bar{x}$ in R^n.

A mapping $T: R^n \to R^m$ is said to be one-to-one if each $\bar{b}$ in R^m is the image of at most one $\bar{x}$ in R^n.

Ex. Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Clear, $\forall\, b \in R^3$, the equation $A\bar{x} = \bar{b}$ is consistent.
Hence T is onto R^3.
Since the equation $A\bar{x} = \bar{b}$ has a free variable, it is not one-to-one.

Theorem. Let $T: R^n \to R^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\bar{x}) = \bar{0}$ has only the trivial solution.

Theorem. Let $T: R^n \to R^m$ be a linear transformation and let $\underline{A}$ be the standard matrix for T. Then:

(a) $\underline{T}$ maps R^n onto R^m if and only if the columns of $\underline{A}$ span R^m.

(b) T is one-to-one if and only if the columns of A are linearly independent.

Ex. Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation.

Sol. $$T(\bar{x}) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Since the matrix columns are linearly independent, T is one-to-one.

TABLE 1 Reflections

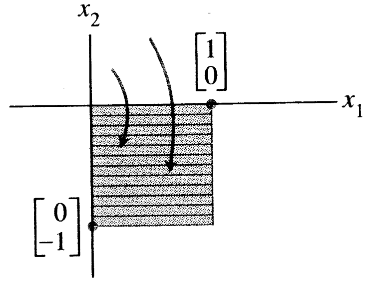
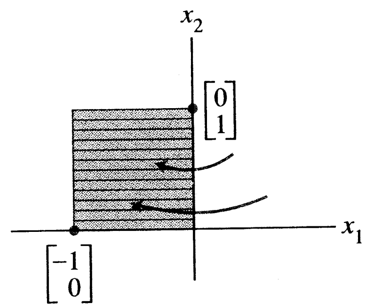
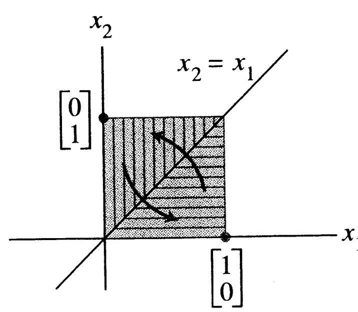
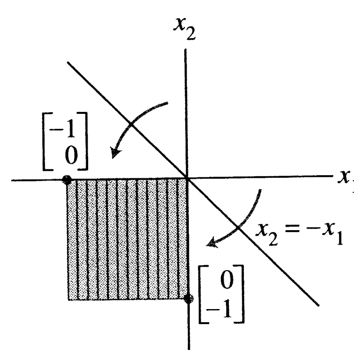
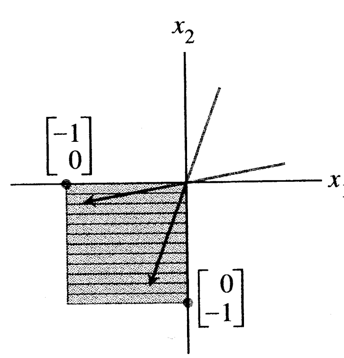
Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Reflection through the x_2-axis		$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$
Reflection through the line $x_2 = -x_1$		$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$
Reflection through the origin		$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$

TABLE 2 Contractions and Expansions

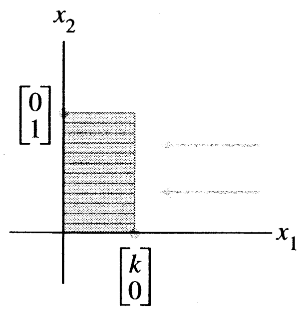
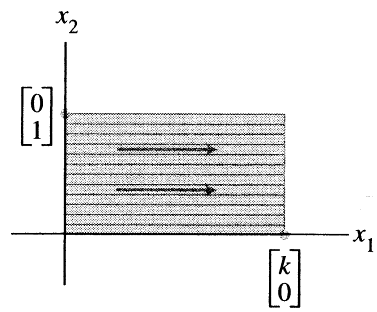
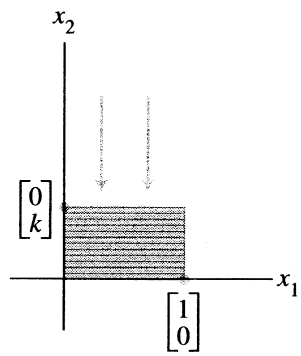
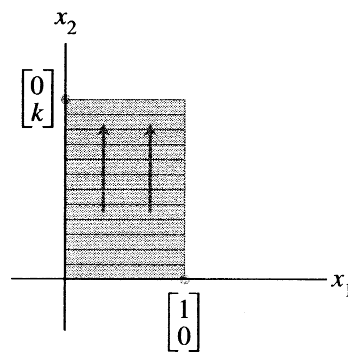
Transformation	Image of the Unit Square		Standard Matrix
Horizontal contraction and expansion	 $0<k<1$	 $k>1$	$\begin{bmatrix}k & 0\\0 & 1\end{bmatrix}$
Vertical contraction and expansion	 $0<k<1$	 $k>1$	$\begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$

TABLE 3 Shears

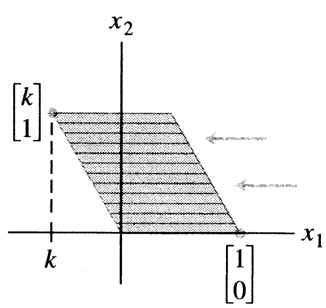
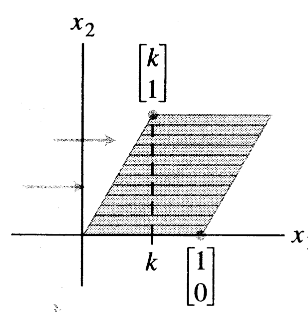
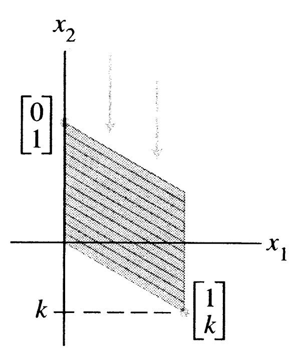
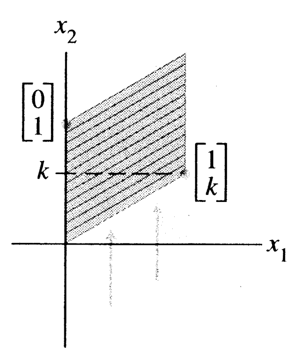
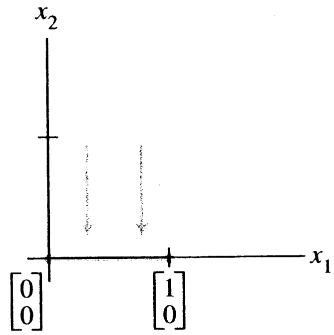
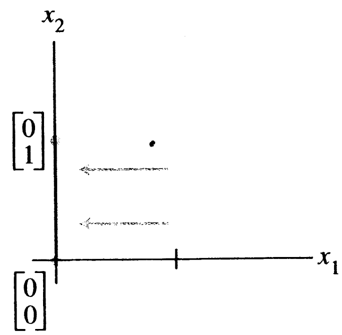
Transformation	Image of the Unit Square		Standard Matrix
Horizontal shear	 $k<0$	 $k>0$	$\begin{bmatrix}1 & k\\0 & 1\end{bmatrix}$
Vertical shear	 $k<0$	 $k>0$	$\begin{bmatrix}1 & 0\\k & 1\end{bmatrix}$

TABLE 4 Projections

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$