

1.8. Introduction to Linear Transformations

A matrix A can be considered as an object that "acts" on a vector \bar{x} to produce a new vector $A\bar{x}$:

$$\bar{x} \longmapsto A\bar{x}$$

Ex:
$$\begin{bmatrix} 4 & -3 & 1 \\ 2 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

\uparrow \uparrow \uparrow
 A \bar{x} $A\bar{x}$

Def. A transformation (or function, or mapping) T from R^n to R^m is a rule that assigns to each vector \bar{x} in R^n a vector $T(\bar{x})$ in R^m .

The set R^n is called the domain of T , and R^m is called the codomain of T .

For \bar{x} in R^n , the vector $T(\bar{x})$ is called the image of \bar{x} .

The set of all images $T(\bar{x})$ is called the range of T .

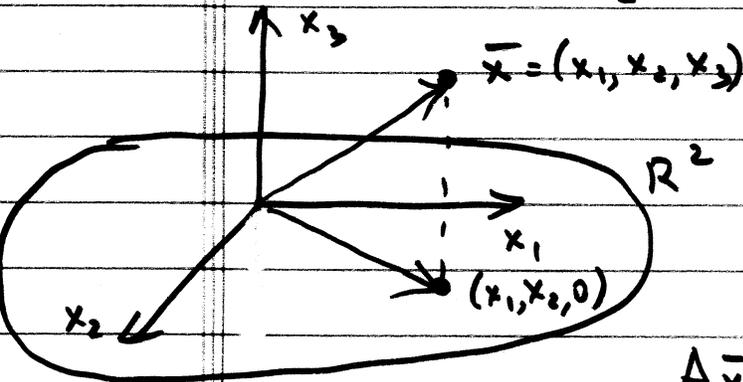
Def. For a $m \times n$ matrix A , the transformation $T(\bar{x}) = A\bar{x}$, $x \in R^n$, is called a matrix transformation.

Ex. Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Then $T(\bar{x}) = A\bar{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$

is a matrix transformation from \mathbb{R}^2 to \mathbb{R}^3 .

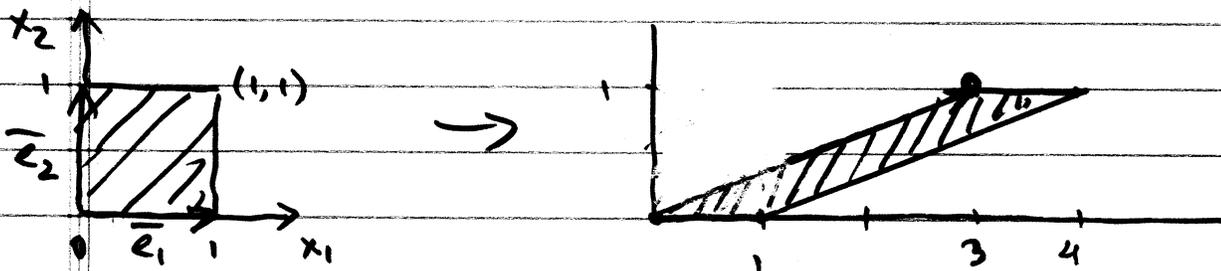
Ex. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ then the transformation $\bar{x} \rightarrow A\bar{x}$ is a projection of \mathbb{R}^3 onto \mathbb{R}^2 .



Indeed, if $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then

$$A\bar{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

Ex. Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(\bar{x}) = A\bar{x}$, $\bar{x} \in \mathbb{R}^2$, is called a shear transformation.



$$A\bar{e}_1 = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 3 \cdot 0 \\ 0 \cdot 1 + 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A\bar{e}_2 = A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Linear Transformations

Def. A transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called linear if

- (i) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^n$
- (ii) $T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

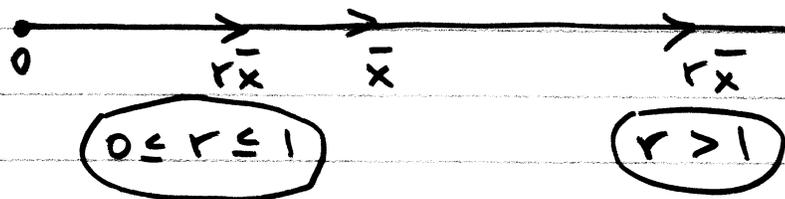
Property of linear transformations:

If $\vec{v}_1, \dots, \vec{v}_p$ are any vectors in \mathbb{R}^n and c_1, \dots, c_p are any scalars, then

$$T(c_1\vec{v}_1 + \dots + c_p\vec{v}_p) = c_1T(\vec{v}_1) + \dots + c_pT(\vec{v}_p).$$

Ex. Given a scalar $r > 0$, define $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(\vec{x}) = r\vec{x}$.

T is called a contraction if $r < 1$, and is called a dilation when $r > 1$.



$$T(\vec{u} + \vec{v}) = r \cdot (\vec{u} + \vec{v}) = r\vec{u} + r\vec{v} = T(\vec{u}) + T(\vec{v})$$

$$T(c\vec{u}) = r \cdot (c\vec{u}) = c(r\vec{u}) = c \cdot T(\vec{u})$$

Ex. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and put $T(\bar{x}) = A\bar{x}$ (4)

Then T is the counterclockwise rotation about the origin through 90° .

Indeed, if $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then

$$T(\bar{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

