

1.7. Linear Independence

Definition. An indexed set of vectors $\{\bar{v}_1, \dots, \bar{v}_p\}$ in \mathbb{R}^n is said to be linearly independent if the vector equation

$$x_1 \bar{v}_1 + x_2 \bar{v}_2 + \dots + x_p \bar{v}_p = \bar{0}$$

has only the trivial solution $x_1 = x_2 = \dots = x_p = 0$.

The set $\{\bar{v}_1, \dots, \bar{v}_p\}$ is said to be linearly dependent if there exist scalars c_1, \dots, c_p , not all zero, such that

$$c_1 \bar{v}_1 + c_2 \bar{v}_2 + \dots + c_p \bar{v}_p = \bar{0}.$$

Example 1. Let $\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \bar{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

a) Determine if the set $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is linearly independent.

b) If possible, find a linear dependence relation among \bar{v}_1, \bar{v}_2 , and \bar{v}_3 .

Solution. a) We must determine if there is a non-trivial solution of the vector equation

$$x_1 \bar{v}_1 + x_2 \bar{v}_2 + x_3 \bar{v}_3 = \bar{0}.$$

Row operations on the associated augmented matrix show that

$$\left[\begin{array}{cccc} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{cccc} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 3 & 6 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{cccc} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right].$$

So, $x_1 = 2x_3$, $x_2 = -x_3$, x_3 is free

Choose any nonzero value for x_3 - say, $x_3 = 5$. Then

$x_1 = 2 \cdot 5 = 10$, $x_2 = -5$, $x_3 = 5$. Therefore,

$$10\bar{v}_1 - 5\bar{v}_2 + 5\bar{v}_3 = \bar{0}.$$

Linear Independence of Matrix Columns

Let an $m \times n$ matrix A be expressed as

$A = [\bar{a}_1 \dots \bar{a}_n]$, where $\bar{a}_1, \dots, \bar{a}_n$ are m -dimensional vectors. Each linear dependence of the form

$$c_1\bar{a}_1 + c_2\bar{a}_2 + \dots + c_n\bar{a}_n = \bar{0}$$

corresponds to a nontrivial solution of the matrix equation $A\bar{x} = \bar{0}$. This argument give the following

Proposition. The columns of a matrix A are linearly independent if and only if the equation $A\bar{x} = \bar{0}$ has only the trivial solution.

Example 2. Determine if the columns of the matrix

$$A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$$

are linearly independent.

Solution. To study $A\bar{x}=\bar{0}$, row reduce the augmented matrix:

$$\begin{aligned} \left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{array} \right] &\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 5 & 8 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{array} \right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]. \end{aligned}$$

So, $x_1 = x_2 = x_3 = 0$ is the only solution of $A\bar{x}=\bar{0}$.

Consequently, the columns of A are linearly independent.

Linear Independence of Two or More Vectors

Theorem 7 (Characterization of Linearly Independent Sets) An indexed set $S = \{\bar{v}_1, \dots, \bar{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

(4)

For instance, if the set $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is linearly dependent and $c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{0}$ such that $c_1 \neq 0$, then $\bar{v}_1 = -\frac{c_2}{c_1}\bar{v}_2 - \frac{c_3}{c_1}\bar{v}_3$ is a linear combination of \bar{v}_2 and \bar{v}_3 .

Theorem 8. If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\bar{v}_1, \dots, \bar{v}_p\}$ in R^n is linearly dependent if $p > n$.

Example 5. The vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ are linearly dependent.

Theorem 9. If a set $S = \{\bar{v}_1, \dots, \bar{v}_p\}$ in R^n contains the zero vector, then the set is linearly dependent.

Proof. Suppose $\bar{v}_1 = \bar{0}$. Then the equality

$$1\bar{v}_1 + 0\bar{v}_2 + \dots + 0\bar{v}_p = \bar{0}$$

shows that S is linearly dependent.

Example 6. Determine by inspection if the given set is linearly dependent.

a) $\begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$; b) $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}$.

Solution.

- a) The set contains four vectors from \mathbb{R}^3 . Theorem 8 shows that the set is linearly dependent.
- b) The set contains the zero vector. Theorem 9 implies that the set is linearly dependent.