

## 1.5. Solution sets of Linear Systems

### Homogeneous Linear Systems

A system of linear equations  $A\bar{x} = \bar{0}$  is called homogeneous.

Such a system always has a solution:  $\bar{x} = \bar{0}$ , called the trivial solution.

For a given equation  $A\bar{x} = \bar{0}$ , the important question is whether there exists a nontrivial solution, that is, a nonzero vector  $\bar{x}$  that satisfies  $A\bar{x} = \bar{0}$ .

Theorem. The homogeneous equation  $A\bar{x} = \bar{0}$  has a nontrivial solution if and only if the equation has at least one free variable.

Example. Determine if the following homogeneous system has a nontrivial solution. Describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0. \end{aligned}$$

$$\left[ \begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{aligned}x_1 - \frac{4}{3}x_3 &= 0 \\x_2 &= 0 \\0 &= 0\end{aligned}$$

$x_3$  is a free variable,  $x_2 = 0$ ,  $x_1 = \frac{4}{3}x_3$ .  
The general solution of  $A\bar{x} = \bar{0}$  is

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}.$$

As a result each solution of  $A\bar{x} = \bar{0}$  is a multiple of the vector

$$\bar{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix},$$

i.e., the solution set is a line through  $\bar{0}$ .

Example. Describe the solution set of the system

$$10x_1 - 3x_2 - 2x_3 = 0.$$

$x_2$  and  $x_3$  are free variables and  $x_1 = .3x_2 + .2x_3$ .  
The general solution is

$$\begin{aligned}\bar{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} .2x_3 \\ 0 \\ x_3 \end{bmatrix} = \\ &= x_2 \begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix},\end{aligned}$$

it is the span of two noncollinear vectors, i.e.,  
the general solution is a plane through  $\bar{0}$ .

## Parametric Vector Form

- 1)  $\bar{x} = t\bar{v}$ ,  $t \in \mathbb{R}$ , is a parametric equation of the line through  $\bar{o}$  ( $\bar{v} \neq \bar{0}$ )
- 2)  $\bar{x} = s\bar{u} + t\bar{v}$ ,  $s, t \in \mathbb{R}$ , is a parametric equation of the plane through  $\bar{o}$ . (both  $\bar{u} \neq \bar{0}$  and  $\bar{v} \neq \bar{0}$ )

## Solutions of Nonhomogeneous Systems

Ex. Describe all solutions of the system  $A\bar{x} = \bar{b}$ , where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}.$$

Solution Row operations on  $[A \bar{b}]$  produce

$$\left[ \begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{aligned} x_1 - \frac{4}{3}x_3 &= -1 \\ x_2 &= 2 \\ 0 &= 0 \end{aligned}$$

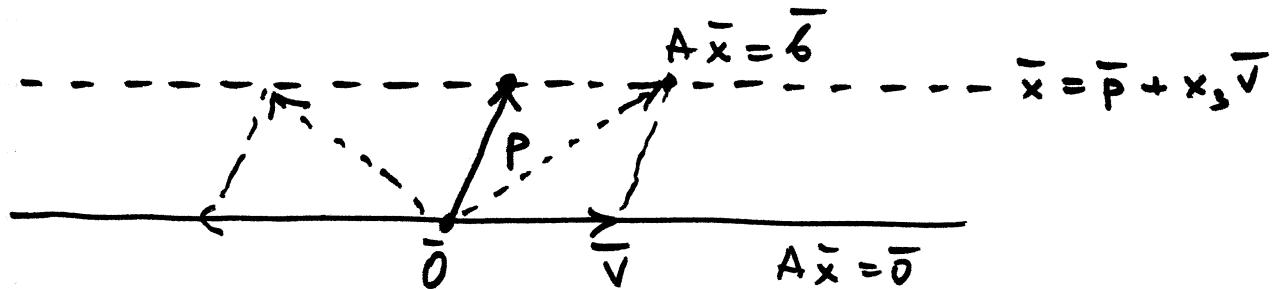
Hence the general solution of  $A\bar{x} = \bar{b}$  has the form

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}.$$

- what is this?

Put  $\bar{p} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ ,  $\bar{v} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$ . Then  $\bar{x} = \bar{p} + x_3 \bar{v}$ ,  $x_3 \in \mathbb{R}$ . (4)

The solution set of  $A\bar{x} = \bar{b}$  is a line through  $\bar{p}$  parallel to the line  $x_3 \bar{v}$ ,  $x_3 \in \mathbb{R}$ .



Theorem. Suppose the equation  $A\bar{x} = \bar{b}$  has a solution for a given vector  $\bar{b}$ , and let  $\bar{p}$  be any particular solution of  $A\bar{x} = \bar{b}$ .

Then the solution set of  $A\bar{x} = \bar{b}$  is the set of all vectors of the form  $\bar{w} = \bar{p} + \bar{v}$ , where  $\bar{v}$  is any solution of the homogeneous equation  $A\bar{x} = \bar{0}$ .