1.4. The Matrix Equation $A\bar{x} = \bar{b}$

A fundamental idea in linear algebra is to view a linear combination of vectors as the product of a matrix and a vector.

Definition. If A is an $m \times n$ matrix, with columns $\bar{a}_1, \dots, \bar{a}_n$, and if $\bar{x}$ is in $\mathbb{R}^n$, then the product of A and $\bar{x}$, denoted by $A\bar{x}$, is the linear combination of the columns of A using the corresponding entries in $\bar{x}$ as weights:

$$A\bar{x} = [\bar{a}_1 \ \dots \ \bar{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\bar{a}_1 + \dots + x_n\bar{a}_n .$$

Ex. 2×3, 3×1

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Ex. For vectors $\bar{v}_1, \bar{v}_2, \bar{v}_3$ in $\mathbb{R}^m$, write the linear combination $3\bar{v}_1 - 5\bar{v}_2 + 7\bar{v}_3$ as a matrix times a vector.

$$3\bar{v}_1 - 5\bar{v}_2 + 7\bar{v}_3 = [\bar{v}_1 \ \bar{v}_2 \ \bar{v}_3] \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}$$

This approach enables us to write a system of linear equations as a *matrix equation*.

Namely,

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned}$$

can be written as

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Theorem. If A is a $m \times n$-matrix, with columns $\bar{a}_1, \ldots, \bar{a}_n$, and if $\bar{b}$ is a vector in R^m, the matrix equation

$$A\bar{x} = \bar{b}$$

has the same solution set as the vector equation

$$x_1 \bar{a}_1 + x_2 \bar{a}_2 + \ldots + x_n \bar{a}_n = \bar{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\,\bar{a}_1 \; \bar{a}_2 \ldots \bar{a}_n \; \bar{b}\,].$$

Existence of Solutions

Theorem. The equation $A\bar{x} = \bar{b}$ has a solution if and only if $\bar{b}$ is a linear combination of the columns of A.

Ex. Let
$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \text{ and } \bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Does the equation $A\bar{x} = \bar{b}$ have a solution for all possible b_1, b_2, b_3?

Solution. Row reduce the augmented matrix for $A\bar{x} = \bar{b}$:

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

Hence $A\bar{x} = \bar{b}$ has no solution if and only if $b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \neq 0$.

In the next theorem, the sentence "The columns of A span R^m" means that every $\bar{b}$ in R^m is a linear combination of the columns of A.

In general, a set of vectors $\{\bar{v}_1, \dots, \bar{v}_p\}$ in R^m spans (or generates) R^n, if $\text{span}\{\bar{v}_1, \dots, \bar{v}_p\} = R^n$.

Theorem 4. Let A be an $m \times n$-matrix. The following statements are logically equivalent.

a. For each $\bar{b}$ in R^m, the equation $A\bar{x} = \bar{b}$ has a solution.

b. Each $\bar{b}$ in R^m is a linear combination of the columns of A

c. The columns of A span R^m.

d. A has a pivot position in every row.

Computation of $A\bar{x}$

Row-Vector Rule. If the product $A\bar{x}$ is defined then the ith entry in $A\bar{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\bar{x}$.

Ex.

$$\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 2\cdot 4 + (-3)\cdot 7 \\ 8\cdot 4 + 0\cdot 7 \\ (-5)\cdot 4 + 2\cdot 7 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}.$$

Properties of the Matrix-Vector Product $A\bar{x}$

If A is an $m \times n$ matrix, $\bar{u}$ and $\bar{v}$ are vectors in R^n, and c is a scalar, then

(a) $A(\bar{u}+\bar{v}) = A\bar{u} + A\bar{v}$

(b) $A(c\bar{u}) = cA\bar{u}$.

Proof. For simplicity, take $n=3$:

$$A = [\bar{a}_1 \ \bar{a}_2 \ \bar{a}_3], \quad \bar{u}, \bar{v} \in R^3$$

If $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ then

$$A(\bar{u}+\bar{v}) = [\bar{a}_1 \ \bar{a}_2 \ \bar{a}_3] \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix} =$$

$$= (u_1+v_1)\bar{a}_1 + (u_2+v_2)\bar{a}_2 + (u_3+v_3)\bar{a}_3 =$$

$$= (u_1\bar{a}_1 + u_2\bar{a}_2 + u_3\bar{a}_3) + (v_1\bar{a}_1 + v_2\bar{a}_2 + v_3\bar{a}_3) =$$

$$= A\bar{u} + A\bar{v}.$$

Similarly, $A(c\bar{u}) = [\bar{a}_1\ \bar{a}_2\ a_3]\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} =$

$= (cu_1)\bar{a}_1 + (cu_2)\bar{a}_2 + (cu_3)\bar{a}_3 =$

$= c\,(u_1\bar{a}_1 + u_2\bar{a}_2 + u_3\bar{a}_3) = c\,A\bar{u}.$