

1.3. Vector Equations

Having a system of linear equations, like

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \end{cases}$$

we were considering the matrix (augmented matrix)

$$\left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \end{array} \right]$$

Another approach uses vector notation. We write symbolically

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$$

This leads us to the notion of a vector.

Vectors in \mathbb{R}^2

A vector is a matrix with only one column.

$$\text{Ex: } \bar{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} .2 \\ .3 \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

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The set of vectors of the form $\begin{bmatrix} a \\ b \end{bmatrix}$ is denoted by R^2 . We will see later that R^2 can be identified with a usual plane.

Def. Two vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ are equal if $a = c$ and $b = d$.

Given two vectors \bar{u} and \bar{v} in R^2 , their sum is the vector $\bar{u} + \bar{v}$ obtained by adding corresponding entries of \bar{u} and \bar{v} .

Ex. $\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} 1+2 \\ -2+5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

Given a vector \bar{u} and a real number c , the scalar multiple of \bar{u} by c is the vector $c\bar{u}$ obtained by multiplying each entry in \bar{u} by c .

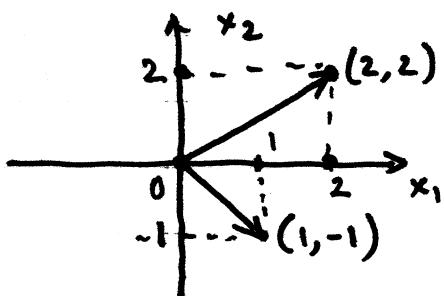
Ex. If $\bar{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $c = 5$, then $c\bar{u} = 5 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$.

The number c above is called a scalar.

Problem. Given $\bar{u} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$. Find $4\bar{u}$, $(-3)\cdot\bar{v}$, $4\bar{u} + (-3)\bar{v}$.

Geometric Description of \mathbb{R}^2

Consider a rectangular coordinate system in the plane.

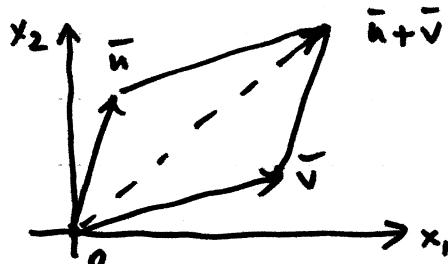


Then we can identify each point (a, b) in the plane with the column vector $\begin{bmatrix} a \\ b \end{bmatrix}$.

The geometric visualization of a vector such as $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is often aided by including an arrow from the origin $(0, 0)$ to $(1, -1)$.

The following rules can be verified by analytic geometry.

Parallelogram Rule for Addition If \bar{u} and \bar{v} in \mathbb{R}^2 are the vectors, then $\bar{u} + \bar{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\bar{0}$, \bar{u} , and \bar{v} .



Ex. Find $\bar{u} + \bar{v}$ for $\bar{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$.

Ex. Let $\bar{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Display \bar{u} , $2\bar{u}$, $-\frac{3}{2}\bar{u}$.

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Vectors in R^3 are 3×1 column matrices that can be identified with the points of 3-space.

Vectors in R^n . If n is a positive integer, R^n denotes the collection of all lists of n real numbers, usually written as $n \times 1$ column matrices

$$\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

The zero vector is $\bar{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Two vectors are equal if their corresponding entries are equal:

if $\bar{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ then $\bar{u} = \bar{v}$ if and only if $u_1 = v_1, \dots, u_n = v_n$.

We define the sum $\bar{u} + \bar{v}$ and the product $c\bar{u}$ by

$$\bar{u} + \bar{v} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}, \quad c\bar{u} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix}.$$

Algebraic Properties of vectors in R^n

For all \bar{u}, \bar{v} , and \bar{w} in R^n , all scalars c and d

- (i) $\bar{u} + \bar{v} = \bar{v} + \bar{u}$
- (ii) $(\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w})$
- (iii) $\bar{u} + \bar{0} = \bar{0} + \bar{u} = \bar{u}$
- (iv) $\bar{u} + (-\bar{u}) = -\bar{u} + \bar{u} = \bar{0}$
- (v) $c(\bar{u} + \bar{v}) = c\bar{u} + c\bar{v}$
- (vi) $(c+d)\bar{u} = c\bar{u} + d\bar{u}$
- (vii) $c(d\bar{u}) = (cd)\bar{u}$
- (viii) $1 \cdot \bar{u} = \bar{u}$

Definition. Given vectors $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p$ in R^n and scalars c_1, c_2, \dots, c_p , the vector

$$\bar{y} = c_1 \bar{v}_1 + c_2 \bar{v}_2 + \dots + c_p \bar{v}_p$$

is called a linear combination of $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p$ with weights c_1, c_2, \dots, c_p .

Ex. Let $\bar{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\bar{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$.

Determine whether \bar{b} is a linear combination of \bar{a}_1 and \bar{a}_2 .

Solution. We need to determine the existence of weights, x_1 and x_2 , such that

$$x_1 \bar{a}_1 + x_2 \bar{a}_2 = \bar{b}.$$

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$$x_1 \cdot \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

↑ ↑ ↑
 \bar{a}_1 \bar{a}_2 \bar{b}

or

$$\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

or

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \\ -5x_1 + 6x_2 &= -3 \end{aligned}$$

We solve the system by row reducing the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right]$$

Hence $x_1 = 3$ and $x_2 = 2$, and $3\bar{a}_1 + 2\bar{a}_2 = \bar{b}$.

We can write the augmented matrix above as

$$[\bar{a}_1 \ \bar{a}_2 \ \bar{b}]$$

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Theorem. A vector equation $x_1\bar{a}_1 + x_2\bar{a}_2 + \dots + x_n\bar{a}_n = \bar{b}$ has the same solution set as the linear system whose augmented matrix is $[\bar{a}_1 \ \bar{a}_2 \ \dots \ \bar{a}_n \ \bar{b}]$

Def. If $\bar{v}_1, \dots, \bar{v}_p$ are in \mathbb{R}^n , then the set of all linear combinations of $\bar{v}_1, \dots, \bar{v}_p$ is denoted by $\text{Span}\{\bar{v}_1, \dots, \bar{v}_p\}$ and is called the subset of \mathbb{R}^n spanned (or generated) by $\bar{v}_1, \dots, \bar{v}_p$.

That is, $\text{Span}\{\bar{v}_1, \dots, \bar{v}_p\}$ is the collection of all vectors that can be written as

$$c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_p\bar{v}_p$$

with c_1, c_2, \dots, c_p scalars.

Asking whether a vector \bar{b} is in $\text{Span}\{\bar{v}_1, \dots, \bar{v}_p\}$ amounts to ask whether the vector equation

$$x_1\bar{v}_1 + x_2\bar{v}_2 + \dots + x_p\bar{v}_p = \bar{b}$$

has a solution.

A Geometric Description of $\text{Span}\{\bar{v}\}$ and $\text{Span}\{\bar{u}, \bar{v}\}$.

If \bar{v} is a nonzero vector in \mathbb{R}^3 , then $\text{Span}\{\bar{v}\}$ is the line through \bar{v} and $\bar{0}$.

If \bar{u}, \bar{v} are nonzero vectors in \mathbb{R}^3 and \bar{v} is not a multiple of \bar{u} , then $\text{Span}\{\bar{u}, \bar{v}\}$ is the plane through $\bar{0}, \bar{u}, \bar{v}$.