

1.2. Row Reduction and Echelon Forms

In the definitions that follow, a nonzero row or column in a matrix means a row or a column that contains at least one nonzero entry; a leading entry of a row refers to the left-most nonzero entry.

Definition. A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

Ex. $\begin{bmatrix} 3 & -1 & 5 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & -1 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 7 & -3 & 4 & 6 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Def. A rectangular matrix is in reduced echelon form if it is in echelon form and satisfies the following additional conditions:

4. The leading entry in each nonzero row is 1.
5. Each leading entry (1) is the only nonzero entry in a column.

Ex.

$$\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Def. An echelon matrix (respectively, reduced echelon matrix) is one that is in echelon (respectively, reduced echelon) form.

Th. Any matrix is row equivalent to one and only one reduced echelon matrix.

Corollary. The leading entries are always in the same positions in any echelon form obtained from a given matrix.

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Def. A pivot position in a matrix \underline{A} is a location in A that corresponds to a leading 1 in the reduced echelon form of A .

A pivot column is a column of A that contains a pivot position.

Ex. Row reduce the matrix A below to echelon form, and locate the pivot columns of A .

$$A = \left[\begin{array}{cccccc} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array} \right] \sim \left[\begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right] \xrightarrow{+1} \sim \left[\begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right] \xrightarrow{+2} \sim$$

$$\left[\begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{array} \right] \xrightarrow{-5/2} \sim \left[\begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{array} \right] \sim$$

$$\left[\begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

The Row Reduction Algorithm

We illustrate the algorithm by an example.

Ex: Transform the following matrix into reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Step 1. Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

↑
Pivot column

Step 2. Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

← pivot

$$\begin{bmatrix} 3 & -1 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 3. Use row replacement operations to create zeros in all positions below the pivot.

$$+(-1)[1] \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 4 Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1-3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

pivot

$$+(-\frac{3}{2})[2] \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

pivot

Step 5. Beginning with the right most pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

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$$\left[\begin{array}{cccccc} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right] + (-6) \cdot [3] \\ + (-2) \cdot [3]$$

$$\left[\begin{array}{cccccc} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right] \times \frac{1}{2}$$

$$\left[\begin{array}{cccccc} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right] + 9 \cdot [2]$$

$$\left[\begin{array}{cccccc} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right] \times \frac{1}{3}$$

→ This is a reduced echelon matrix.

Solutions of Linear Systems

Ex. Consider the linear system

$$\begin{aligned} x_1 + 6x_2 + 2x_3 - 5x_4 - 2x_5 &= -4 \\ 2x_3 - 8x_4 - x_5 &= 3 \\ x_5 &= 7 \end{aligned}$$

Form the augmented matrix for the system:

$$\left[\begin{array}{ccccc} \frac{1}{6} & 2 & -5 & -2 & -4 \\ 0 & 0 & \frac{3}{2} & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 7 \end{array} \right]$$

Performing elementary row operations, we get the reduced echelon form:

$$\left[\begin{array}{ccccc} \frac{1}{6} & 0 & 3 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 7 \end{array} \right].$$

The associated system now is

$$\begin{aligned} x_1 + 6x_2 + 3x_4 &= 0 \\ x_3 - 4x_4 &= 5 \\ x_5 &= 7 \end{aligned}$$

The variables x_1, x_3, x_5 , corresponding to pivot columns in the matrix are called basic variables. The remaining variables x_2 and x_4 are called free variables.

We solve the system above for the basic variables in terms of the free variables, as follows:

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$$x_1 = -6x_2 - 3x_4$$

x_2 is free

$$x_3 = 5 + 4x_4$$

x_4 is free

$$x_5 = 7$$

This is the general solution for the system
 We see that free variables, x_2 and x_4 , can have any value, and x_1, x_3 depend on x_2, x_4 .

Existence and Uniqueness of Solutions

Th. A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column

$$\begin{bmatrix} 0 & \dots & 0 & b \end{bmatrix}$$

If a linear system is consistent, then the solution set contains either

- (i) a unique solution, when there are no free variables,
- (ii) infinitely many solutions, when there is at least one free variable.

How to find and describe all solutions of a linear system?

Using row reduction to solve a linear system

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.