

1.1. Systems of Linear Equations

Def. A linear equation in the variables x_1, \dots, x_n is an equation of the form

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b,$$

where the coefficients a_1, a_2, \dots, a_n and b are real or complex numbers, known in advance, and x_1, x_2, \dots, x_n are unknown numbers.

Ex. $4x_1 - 5x_2 + x_3 = 6,$

$$\sqrt{2}x_1 - \sqrt{3}x_3 = 1.$$

Def. A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same variables — say, x_1, \dots, x_n .

Ex. $2x_1 - x_2 + 5x_3 = 8$
 $x_1 - 4x_3 = -7$

Def. A solution of the linear system is a list (s_1, s_2, \dots, s_n) of numbers that makes each equation a true statement when the values s_1, \dots, s_n are substituted for x_1, \dots, x_n , respectively.

Ex. $(3, -2)$ is a solution of the system

$$\begin{aligned}4x_1 + 5x_2 &= 2 \\-x_1 + 2x_2 &= -7\end{aligned}$$

Indeed,

$$\begin{aligned}4 \cdot 3 + 5 \cdot (-2) &= 2 \\(-1) \cdot 3 + 2 \cdot (-2) &= -7.\end{aligned}$$

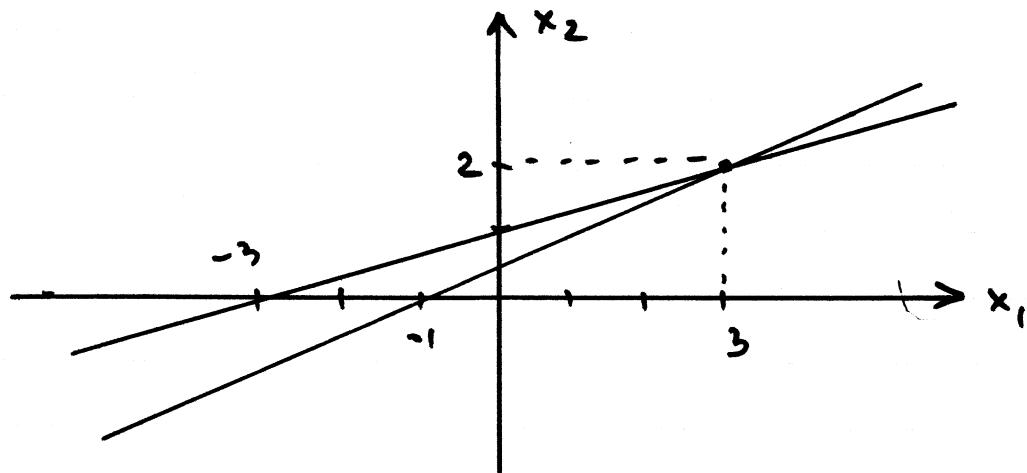
Def. The set of all possible solutions of a given linear system is called the solution set of the system.

Two linear systems are called equivalent if they have the same solution set.

Graphical interpretation of the solution set.

Case 1.

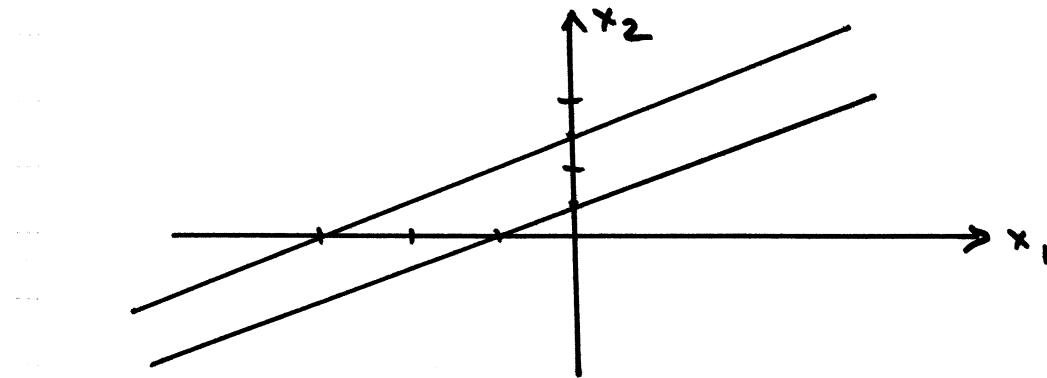
$$\begin{aligned}x_1 - 2x_2 &= -1 \\-x_1 + 3x_2 &= 3\end{aligned}$$



The linear system has exactly one solution, $(3, 2)$.

Case 2.

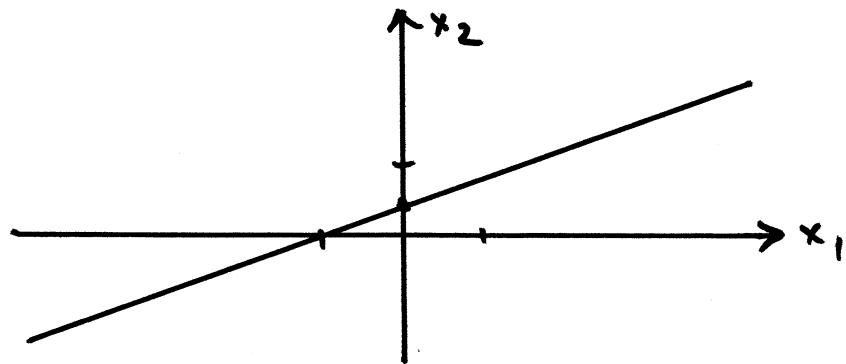
$$\begin{aligned}x_1 - 2x_2 &= -1 \\-x_1 + 2x_2 &= 3\end{aligned}$$



The linear system has no solutions.

Case 3.

$$\begin{aligned}x_1 - 2x_2 &= -1 \\-x_1 + 2x_2 &= 1\end{aligned}$$



The linear system has infinitely many solutions.

Theorem. Any system of linear equations in n variables x_1, \dots, x_n has either

1. no solutions, or
2. exactly one solution, or
3. infinitely many solutions.

Matrix Notation

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The essential information of a linear system can be recorded compactly in a rectangular table, called a matrix.

Ex. Given the system

$$\begin{aligned}x_1 - 2x_2 + x_3 &= 0 \\2x_2 - 8x_3 &= 8 \\-4x_1 + 5x_2 + 9x_3 &= -9,\end{aligned}$$

the table

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

is called the coefficient matrix (or matrix of coefficients) of the system.

The table

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

is called the augmented matrix of the system.

Def. The size of a matrix tells how many rows and columns it has. If m and n are positive integers, an $m \times n$ matrix is a rectangular table with m rows and n columns.

Note: The number of rows always comes first!

Solving a linear System

This section describes an algorithm for solving linear system. The basic strategy here is to replace one system with an equivalent system that is easier to solve.

There are three basic operations to simplify a linear system. These operations are called elementary equation operations.

1. (Replacement) Replace one equation by the sum of itself and a multiple of another equation.
2. (Interchange) Interchange two equations.
3. (Scaling) Multiply all coefficients (entries) in an equation by a nonzero constant.

Ex. Solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \quad \left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right]$$

Solution. 1) Keep x_1 in the first equation and eliminate it from the other equations. To do this, multiply the first equation by $\frac{1}{4}$ and add to equation 3.

$$\begin{aligned}x_1 - 2x_2 + x_3 &= 0 \\2x_2 - 8x_3 &= 8 \\-3x_2 + 13x_3 &= -9\end{aligned}$$

$$\left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

2) Use x_2 in equation 2 to eliminate $-3x_2$ in equation 3.

$$\begin{aligned}x_1 - 2x_2 + x_3 &= 0 \\x_2 - 4x_3 &= 4 \\-3x_2 + 13x_3 &= -9\end{aligned}$$

$$\left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right]$$



$$\begin{aligned}x_1 - 2x_2 + x_3 &= 0 \\x_2 - 4x_3 &= 4 \\x_3 &= 3\end{aligned}$$

$$\left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

Observe that a new system has a triangular form.

Finally, using the way back, we get

$$\begin{aligned}x_1 &= 29 \\x_2 &= 16 \\x_3 &= 3\end{aligned}$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

Th. If a system of linear equations is obtained from another linear system by using elementary equation operations, then both systems are equivalent, i.e., they have the same solution set.

The example above illustrates how operations on equations in a linear system correspond to operations on the appropriate rows of the augmented matrix.

This gives us an idea that, while solving a system, we can deal with the augmented matrix only.

The three basic equation operations above correspond to the following operations on the augmented matrix.

Elementary Row Operations

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Def. We say that two matrices are row equivalent if there is a sequence of elementary row operations that transform one matrix into the other.

Th. Two linear systems have the same solution set if and only if their augmented matrices are row equivalent.

Existence of a Solution

Def. A system of linear equations is said to be consistent if it has a solution.
A system is inconsistent if it has no solution.

Ex. Determine if the system is consistent:

$$\begin{aligned}x_1 - 2x_2 + x_3 &= 0 \\2x_2 - 8x_3 &= 8 \\-4x_1 + 5x_2 + 9x_3 &= -9\end{aligned}$$

Sol. Performing elementary operations, we get

$$x_1 - 2x_2 + x_3 = 0$$

$$x_2 - 4x_3 = 4$$

$$x_3 = 3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

The system is consistent.

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Ex: Determine if the system is consistent:

$$x_2 - 4x_3 = 8$$

$$2x_1 - 3x_2 + 2x_3 = 1$$

$$5x_1 - 8x_2 + 7x_3 = 1$$

Sol.

$$\left[\begin{array}{cccc} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array} \right] \xrightarrow{\quad} \left[\begin{array}{cccc} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array} \right] + \left(-\frac{5}{2} \right) [1] \rightarrow \left[\begin{array}{cccc} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array} \right]$$

$$\rightarrow \rightarrow \left[\begin{array}{cccc} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{array} \right].$$

$$+ \frac{1}{2} \cdot [2]$$

Going back to the system, we get

$$2x_1 - 3x_2 + 2x_3 = 1$$

$$x_2 - 4x_3 = 8$$

$$0 \cdot x_3 = \frac{5}{2}$$

Since there is no value of x_3 to make the equation

$$0 \cdot x_3 = \frac{5}{2}$$

true, we conclude that the system is inconsistent.

General Rule

A system of linear equations is inconsistent if and only if its augmented matrix is row equivalent to a matrix that has a row consisting of all zeros besides the last entry.