Convex Sets

AMS Special Session

April 14-15, 2007, Hoboken, New Jersey

Organizers: David Larman, University College London
Valeriu Soltan, George Mason University

PROGRAM OF THE SESSION

Saturday, April 14

8:30 am	R. Schneider. *Stability results for convex bodies.*
9:00 am	D. A. Klain. *Isometry invariant valuations on hyperbolic space.* (cancelled)
9:30 am	M. Ludwig. *Bivaluations on convex bodies.*
10:00am	D. E. Pallaschke*, R. Urbański. *Pairs of compact convex sets.*
10:30am	H. Martini, Z. Mustafaev.* *Some applications of cross-section measures in Minkowski spaces.*
2:30 pm	J. Pach*, G. Tóth. *Decomposition of multiple coverings into coverings.*
3:00 pm	P. Brass. *Distributing an infinite point sequence uniformly in a region.*
3:30 pm	R. Connelly*, J.-M. Schlenker. *On the infinitesimal rigidity of weakly convex polyhedra.* (cancelled)
4:00 pm	G. Fejes Tóth. *The moment theorem for convex sets.*
4:30 pm	T. Bisztriczky. *Classification of bicyclic 4-polytopes.*
5:00 pm	A. Bezdek. *On iterative processes generating dense point sets.*

Sunday, April 15

8:30 am	E. Lutwak, D. Yang, G. Zhang.* *Volume inequalities of L_p projection bodies.*
9:00 am	R. Howard*, D. Ryabogin, A. Rusu, A. Zvavitch. *Determining symmetric convex bodies by the perimeters of their central sections.*
9:30 am	P. Goodey, M. Kiderlen, W. Weil.* *Weighted projection means of convex bodies.*
10:00am	I. Bárány*, A. Hubard, J. Jesus. *Slicing convex sets and measures by a hyperplane.*
10:30am	G. Pataki. *On the closedness of the linear image of a closed convex cone.*
2:30 pm	D. G. Larman*, N. Garcia-Colin. *Radon-type results arising from McMullen's projective set problem.*
3:00 pm	J. Alonso*, P. Martín. *Characterizations of ellipsoids by sections.*
3:30 pm	E. Schulte. *Combinatorial multihedrality in tilings.*
4:00 pm	M. Rodriguez Álvarez*, J. V. Pérez. *Evenly convexity.*
4:30 pm	V. Soltan. *Characteristic properties of convex quadric surfaces.*

ABSTRACTS

Javier Alonso*, Pedro Martín. Universidad de Extremadura, Badajoz, Spain.

Characterizations of ellipsoids by sections.

Let S be the boundary of a convex body in the d-dimensional Euclidean space $\mathbb{E}^d$ $(d \geq 3)$. It is well known that S is an ellipsoid if and only if the section of S given by any hyperplane is ellipsoidal. The question of whether it is actually necessary to consider "any" hyperplane to characterize S as ellipsoid or is enough to consider "some" hyperplanes is at the origin of an important family of characterizations of ellipsoids. In that context, we study whether we can restrict the hyperplanes to those that are parallel to two or three fixed hyperplanes and also whether we can consider only hyperplanes that contain one of two fixed linear varieties.

Imre Bárány*, Alfredo Hubard, Jerónimo Jesus. Alfréd Rényi Institute of Mathematics, Budapest, Hungary.

Slicing convex sets and measures by a hyperplane.

Convex bodies $K_1, \ldots, K_d \subset \mathbb{R}^d$ are said to be well separated if $\text{aff}\{x_1, \ldots, x_d\}$ is a nondegenerate hyperplane for every $x_1 \in K_1, \ldots, x_d \in K_d$. The main result in this talk says that if $K_1, \ldots, K_d$ are well separated convex bodies in $\mathbb{R}^d$ and $\alpha_1, \ldots, \alpha_d \in [0, 1]$, then there exists a unique oriented halfspace, H, such that $|H \cap K_i| = \alpha_i |K_i|$ for every $i = 1, \ldots, d$, where $|K|$ denotes the volume of the convex body K. The result is extended from convex bodies to measures.

András Bezdek. Auburn University, Auburn, AL.

On iterative processes generating dense point sets.

There are several results in the literature concerning iterative processes in the plane. A typical problem starts with the description of a geometric construction, which when applied to an initial point set generates larger point sets. The problem usually is to prove that repeated expansions lead to an everywhere dense point set. We refer to D. Ismailescu, who started to investigate the construction "add the circumcenters (CC) (incenters (IC), orthocenters (OC) respectively) of all nondegenerate triangles formed by existing points." In a joint paper Iorio, Ismailecsu, Radoicic and Silva solved the planar IC and the planar CC problem and stated conjectures concerning the planar OC problem. The talk outlines the solutions of the following versions: - planar OC problem (with G. Ambrus, 2005) - 3-dimensional IC problem (with G. Ambrus, 2006) - hyperbolic and spherical IC problem (with T. Bisztriczky, 2006) - iterative processes in lattices (2007).

Ted Bisztriczky. University of Calgary, Calgary, Canada.

Classification of bicyclic 4-polytopes.

Bicyclic 4-polytopes were introduced by Z. Smilansky in 1990, and they are the convex hulls of a finitely many evenly spaced points on the generalized trigonometric moment curve in real 4-space. In his introduction of these polytopes, Smilansky conjectured that the number of combinatorial types, with n vertices, is at least $[n/4]$, with equality if n is a prime. In this joint work with J. Lawrence, we examine this problem in the context of classifying all cyclically generated 4-polytopes.

Peter Brass. City College of New York, New York, NY.

Distributing an infinite point sequence uniformly in a region.

Given a convex set C, one way to measure how evenly a point set $p_1, \ldots, p_n$ is distributed in C is the ratio of the $\text{inradius}(p_1, \ldots, p_n) = \sup_{q \in C} \inf_{1 \le i \le n} d(q, p_i)$ to the minimum distance in the set. It is easy to see that

$$\text{inradius}(p_1, \ldots, p_n)/\text{mindist}(p_1, \ldots, p_n) \ge 1/\sqrt{3},$$

and sections of the triangular lattice reach that lower bound. In this talk, we look at infinite sequences $p_1, p_2, \ldots$ such that each beginning part $p_1, \ldots, p_n$ is well-distributed in this sense. Specifically, that

$$\lim_{n \to \infty} \sup \big(\text{inradius}(p_1, ..., p_n)/\text{mindist}(p_1, \ldots, p_n)\big)$$

is small. If we construct the point set by refining a triangular lattice section, this lim sup is 1, and the same can be reached from any starting set by the process of Voronoi insertion, as observed recently in a paper by Teramoto, Asano, Katoh and Doerr. This suggests the question whether there is any sequence for which the lim sup is less than one. We give a lower bound of $1/\sqrt{2}$. This is related to a property of the minimum angles of the sequence of Delaunay-triangulations of these sets.

Robert Connelly*, Jean-Marc Schlenker. Cornell University, Ithaca, NY.

On the infinitesimal rigidity of weakly convex polyhedra.

The main motivation here is a question: whether any polyhedron which can be subdivided into convex pieces without adding a vertex, and which has the same vertices as a convex polyhedron, is infinitesimally rigid. We prove that it is indeed the case for two classes of polyhedra: those obtained from a convex polyhedron by "denting" at most two edges at a common vertex, and suspensions with a natural subdivision.

Gábor Fejes Tóth. Alfréd Rényi Institute of Mathematics, Budapest, Hungary.

The moment theorem for convex sets.

We prove the following generalization of the "moment theorem" which was proved previously for the case when R is a convex polygon with at most six sides.

Theorem. Let R be a convex domain and P a set of $n \geq 2$ points in the plane. Let H be a regular hexagon centered at the origin with $area(H) = area(R)/n$. Then we have for any non-increasing function f defined for non-negative reals

$$\int_R f(\min_{p\in P} \|p-x\|)\, dx \leq n \int_H f(\|x\|)\, dx$$

For a non-decreasing function f the inequality stands with the reversed sign.

Paul Goodey, Markus Kiderlen, Wolfgang Weil.* University of Karlsruhe, Karlsruhe, Germany.

Weighted projection means of convex bodies.

The k-th *projection mean body* $P_k(K)$ of a convex body $K \subset \mathbb{R}^d$ is the Minkowski average of the orthogonal projections $P_L(K)$ of K onto all k-dimensional subspaces L in $\mathbb{R}^d$. The operator $K \mapsto P_k(K)$ has been studied by various authors (Schneider, Spriestersbach, Goodey, Jiang, Kiderlen), its injectivity behavior is still not known completely. As a variant, we introduce the k-th *m-weighted projection mean body* $P_{k,m}(K)$ of K by replacing the projection $P_L(K)$ by a weighted projection $P_{L,m}(K)$, $m > -k$, where $P_k(K) = P_{k,\infty}(K)$. Kiderlen (1999) showed that $K \mapsto P_{k,1-k}(K)$ is injective. Here, we settle two further cases by showing that $K \mapsto P_{k,1}(K)$ and $K \mapsto P_{k,2}(K)$ are injective. This follows from more general results on spherical projections and liftings.

Ralph Howard*, Dmitry Ryabogin, Anamaria Rusu, Artem Zvavitch. University of South Carolina, Columbia, SC.

Determining symmetric convex bodies by the perimeters of their central sections.

Let $\mathcal{P}^n_k$ be the collection of all C^1 convex bodies K in $\mathbb{R}^n$ symmetric about the origin with the property that for all k-dimensional linear subspaces P of $\mathbb{R}^n$ $V_{k-1}(P \cap \partial K) = V_{k-1}(P \cap \partial \mathbf{B}^n)$ where $\mathbf{B}^n$ is the Euclidean ball. (That is $K \in \mathcal{P}^n_k$ is a centrally convex body with C^1 boundary and the property that the $(k-1)$-dimensional "parameter" of $P \cap K$ is the same as that of $P \cap \partial \mathbf{B}^n$ for all k-dimensional central sections of K.) We show that in this class the ball is isolated in the sense that all one-parameter analytic deformations of the ball in $\mathcal{P}^n_k$ are constant. This gives evidence to support the conjecture that if K_1 and K_2 are two convex bodies symmetric about the origin whose sections by any k-dimensional plane through the origin have equal perimeters, then $K_1 = K_2$, a question posed by Richard Gardner in his book *Geometric Tomography* in the case $k = 2$ and $n = 3$.

Daniel A. Klain. University of Massachusetts, Lowell, MA.

Isometry invariant valuations on hyperbolic space.

Hyperbolic area is characterized as the unique continuous isometry-invariant simple valuation on convex polygons in the hyperbolic plane. It is also shown that continuous isometry-invariant simple valuations on polytopes in $(2n+1)$-dimensional hyperbolic space for $n \geq 1$

are determined uniquely by their values at ideal simplices. The proofs exploit a connection between valuation theory in hyperbolic space and an analogous theory on the Euclidean sphere. These results lead to characterizations of continuous isometry-invariant valuations on convex polytopes and convex bodies in the hyperbolic plane, a partial characterization in hyperbolic 3-space, and a mechanism for deriving many fundamental theorems of hyperbolic integral geometry, including kinematic formulas, containment theorems, as well as isoperimetric and Bonnesen-type inequalities.

David G. Larman*, Natalia Garcia-Colin. University College London, London, England.

Radon-type results arising from McMullen's projective set problem.

The well known problem of McMullen asks for the largest number v such that any set with v points in general position in d-space can be mapped, by a permissible projective transformation, onto the vertices of a convex polytope. This translates, via Gale diagrams, to finding the smallest number w such that any set X with w points can be partitioned into two sets A, B such that the convex hulls of $A \setminus \{x\}$ and $B \setminus \{x\}$ overlap for all $x \in X$. I conjectured, in 1972, that $v = 2d + 1$ (and hence $w = 2d + 3$). This remains unresolved for $d > 4$. I will discuss several results and conjectures around this problem.

Monika Ludwig. Polytechnic University, New York, NY.

Bivaluations on convex bodies.

A bivaluation $\mu(K, L)$, defined for convex bodies K, L in $\mathbb{R}^n$, is a real valued function that is a valuation in either variable, provided that the other variable is held fixed. So, for example, for a bivaluation μ,

$$\mu(K_1, L) + \mu(K_2, L) = \mu(K_1 \cup K_2, L) + \mu(K_1 \cap K_2, L)$$

holds for convex bodies K_1, K_2, L if $K_1 \cup K_2$ is convex. Mixed volumes provide an import example: $(K, L) \mapsto V(K, ..., K, L)$. We describe classification theorems for bivaluations and an equi-affine characterization theorem for $V(K, ..., K, L)$.

Erwin Lutwak, Deane Yang, Gaoyong Zhang.* Polytechnic University, New York, NY.

Volume inequalities of L_p projection bodies.

Projection body is a fundamental notion in the Brunn-Minkowski theory of convex bodies. The volume inequality for the polar of a projection body, called the Petty projection body, is an affine isoperimetric inequality which has ellipsoids as extremals and is stronger than the Euclidean isoperimetric inequality. Recently, the notion of L_p projection body was introduced with the classical notion as the L_1 case. An L_p affine isoperimetric inequality was proved and found applications to Sobolev inequalities. In this talk, we consider volume inequalities of L_p projection bodies having parallelotopes as extremals.

Horst Martini, Zokhrab Mustafaev.* University of Houston-Clear Lake, Houston, TX.

Some applications of cross-section measures in Minkowski spaces.

We establish the sharp lower bound for the inner radius of the unit ball for the Holmes-Thompson measure using cross-section measures in Minkowski spaces. We also establish the sharp upper bound for the outer radius of the unit ball for the Busemann measure. Furthermore, we give a characterization of ellipsoids in $\mathbb{R}^d$ via codimensional cross-section measures yielding the con.rmation of a conjecture of Rogers and Shephard.

János Pach*, Geza Tóth. City College and CUNY, New York, NY.

Decomposition of multiple coverings into coverings.

Let $m(k)$ denote the smallest positive integer m such that any m-fold covering of the plane with axis-parallel unit squares splits into at least k coverings. We show that $m(k) = O(k^2)$, and generalize this result to translates of any centrally symmetric convex polygons in the place of squares. For unit disks, instead of squares, it is not known whether the corresponding function is finite.

Diethard Ernst Pallaschke*, Ryszard Urbański. University of Karlsruhe, Karlsruhe, Germany.

Pairs of compact convex sets.

Let $X = (X, \tau)$ be a topological vector space and $\mathcal{K}(X)$ the family of all nonempty compact convex subsets of X. Endowed with the Minkowski addition $A + B = \{a + b \mid a \in A, b \in B\}$ the set $\mathcal{K}(X)$ is a commutative semigroup with cancellation property. An equivalence relation on $\mathcal{K}^2(X) = \mathcal{K}(X) \times \mathcal{K}(X)$ is given by $(A, B) \sim (C, D)$ iff $A + D = B + C$ and an ordering by: $(A, B) \leq (C, D)$ iff $A \subset C$ and $B \subset D$. A pair $(A, B) \in \mathcal{K}^2(X)$ is called minimal if there exists no equivalent pair (C, D) with $(C, D) < (A, B)$. In 2-dimensions equivalent minimal pairs are uniquely determined up to translation. This is not longer true for higher dimensions. We consider also minimal pairs of closed bounded sets and minimality under constraints. A separation law for compact convex sets is proved, which is equivalent to the cancellation law. Within the frame of an ordered commutative semigroup, pairs of compact convex sets correspond to fractions and minimal pairs to relative prime fractions.

Gábor Pataki. University of North Carolina, Chapel Hill, NC.

On the closedness of the linear image of a closed convex cone.

One of the most fundamental questions in convex analysis is also the simplest: given a closed convex cone, and a linear mapping, under what conditions is the image of the cone closed? In the literature several simple sufficient conditions are known, but the only known exact characterizations are much more involved. We give a surprisingly simple condition which

1) is necessary for all cones,

2) unifies and generalizes several classic, seemingly disparate conditions, such as an intersecting in the interior type condition, and the polyhedrality of the cone,

3) is necessary and sufficient for a large class that we call nice cones.

Nice cones subsume most cones that occur in optimization, such as the semidefinite cone, cones arising from p-norms, and of course polyhedral cones. The results are applicable in the duality theory of conic systems, and potentially in other areas as well.

Margarita Rodríguez Álvarez*, José Vicente Pérez. University of Alicante, Alicante, Spain.

Evenly convexity.

A set is evenly convex if it is the intersection of some family (possibly empty) of open halfspaces. This class of convex sets was introduced by Fenchel in 1952 in order to extend the polarity theory to nonclosed convex sets. We show that this large class of convex sets captures the most outstanding properties of the subclass of closed convex sets.

Properties of convex sets are often used to study convex and quasiconvex functions because these classes of functions are characterized by the convexity of their epigraphs and sublevel sets, respectively.

In the same way, in 1980, Martínez-Legaz and Passy and Prisman, independently, started to use evenly convex sets in quasiconvex programming de.ning the evenly quasiconvex functions as those having evenly convex sublevel sets. We consider functions with evenly convex epigraphs, the so-called evenly convex functions, and study the main properties of this class of convex functions that contains the important subclass of lower semicontinuous convex functions. In particular, we try separate these two classes of functions and study if the class of evenly convex functions is closed under the main operations.

Rolf Schneider. Universität Freiburg, Freiburg, Germany.

Stability results for convex bodies.

We report on recent stability results for convex bodies, the first two belonging to geometric tomography, and the second to affine inequalities. First, we provide simple data for the determination of non-symmetric bodies by projections or sections, and obtain explicit estimates quantifying the following. If the orthogonal projections of two convex bodies have their mean widths and their Steiner points close together, then the bodies are close to each other. If the hyperplane sections through a common interior point of two convex bodies have their volumes and their centroids close together, then the bodies are close to each other (the latter is joint work with Károly Böröczky Jr.). Second, for a convex body, define the volume quotient as the ratio of the smallest volume of the circumscribed ellipsoids to the largest volume of the inscribed ellipsoids. It attains its maximum if and only if the body is a simplex. We improve this result by estimating the BanachMazur distance of the body from a simplex if its volume quotient is close to the maximum (joint work with Daniel Hug).

Egon Schulte. Northeastern University, Boston, MA.

Combinatorial multihedrality in tilings.

A locally finite face-to-face tiling of Euclidean space by convex polytopes is called combinatorially multihedral (or combinatorially crystallographic, or combinatorially periodic) if its combinatorial automorphism group has only finitely many orbits on the tiles. We describe a local characterization of combinatorial multihedrality of tilings in terms of large enough neighborhood complexes (centered coronas) of tiles. This generalizes the Local Theorem for Monotypic Tilings, which gives necessary and sufficient conditions for combinatorial tile-transitivity. Both results are joint work with Nikolai Dolbilin. We also discuss a combinatorial analogue of aperiodicity of prototile sets.

Valeriu Soltan. George Mason University, Fairfax, VA.

Characteristic properties of convex quadric surfaces.

We prove that the boundary of an n-dimensional closed convex set $B \subset \mathbb{R}^n$, possibly unbounded, is a convex quadric surface if and only if the middle points of every family of parallel chords of B lie in a hyperplane. An auxiliary result states that the boundary of B is a convex quadric surface if and only if there is a point $p \in \operatorname{int} B$ such that all sections of $\operatorname{bd} B$ by 2-dimensional planes through p are convex quadric curves. Generalizations of these statements that involve boundedly polyhedral sets are given.

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