Chaotic itinerancy based on attractors of one-dimensional maps

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A general methodology is described for constructing systems that have a slowly converging Lyapunov exponent near zero, based on one-dimensional maps with chaotic attractors. In certain parameter ranges, these relatively simple systems display the properties of intermittent dynamics known as chaotic itinerancy. We show that in addition to the local sensitivity characteristic of chaotic dynamics, these itinerant systems display a global sensitivity, in the sense that fine-scale additive noise may significantly change the natural measure on the large scale. [DOI: 10.1063/1.1582332]

In this article we build simple examples of systems undergoing chaotic itinerancy, in order to study mechanisms for its creation. Of particular interest in this article is building simplest possible examples of globally sensitive attractors, meaning those that undergo large-scale changes in natural measure as a consequence of fine-scale noise contributions.

I. INTRODUCTION

The study of chaotic itinerancy[1,2] has unified interest in dynamical systems of widely different origins. Although they may come from deterministic or stochastic models, discrete or continuous dynamics, they commonly share several interesting properties that are relatively intractable at present, including nonhyperbolicity, slow convergence of Lyapunov exponents, metastability and intermittency.

Our motivation in this article is to study the mechanism of chaotic itinerancy by constructing the simplest possible examples of such behavior. We will begin with a one-dimensional map with a chaotic attractor, and build a system with an attractor that contains the original attractor. The new system will have a substantial amount of natural measure on the original attractor but have intermittent excursions from it. The repeated alternation between the vicinity of the original attractor and the excursions is one of the most straightforward manifestations of chaotic itinerancy.

A particular aspect of chaotic itinerancy that we find most interesting is that it can be accompanied by *global* sensitivity, by which we mean sensitivity of natural measure to dynamical noise. A familiar property of chaotic systems is *local* sensitivity to initial conditions. By this we mean that by changing the initial condition of a trajectory a small amount, or by adding dynamical noise along the trajectory, the altered trajectory diverges quite rapidly from the original. It is perhaps a surprising fact that it is common for chaotic systems to be globally insensitive, meaning that even with the sensitivity on a trajectory-by-trajectory basis, the chaotic attractor itself (or more precisely, its natural measure) is relatively insensitive. In other words, a small infusion of small-scale noise to a chaotic attractor may cause only small, proportional changes to the natural measure of the resulting stochastic dynamical system.

In this article, we show examples for which the global insensitivity fails. These examples are both locally and globally sensitive to small dynamical noise. Through the mechanism of chaotic itinerancy, we construct deterministic systems for which small additive noise creates disproportionately large changes in the natural measure of the resulting stochastic dynamical system. The size of these changes is controlled by a scaling law whose exponent, called the "hyperbolicity exponent," depends on the rate of convergence of a Lyapunov exponent close to zero. More precisely, this exponent can be expressed in terms of the moments of the probability distribution of finite-time Lyapunov exponents of the underlying deterministic system.

Hyperbolic systems are globally stable. The hyperbolicity exponent is a rough attempt to quantify the extent to which a nonhyperbolic system has hyperbolic-like properties. When this exponent is close to zero, the global stability is lost. The substantial effect of noise on nonhyperbolic systems has been studied earlier from several different perspectives. In such systems, noise causes unshadowable trajectories,[3] trajectory deviations from the attractor,[4] and slow relaxation to the natural measure.[5]

Of course, small noise added to an attractor near a crisis or other global bifurcation can cause the destruction of a basin, for example as an unstable manifold crosses a stable manifold forming the basin boundary. This may result in an abrupt change in natural measure, as a function of noise level. Here we are interested in more continuous changes of natural measure, ones that are not caused by destruction of attractor or basin but by more subtle phenomena.

II. CONSTRUCTION OF EXAMPLES

Let $g: R \to R$ be a one-dimensional map and define a new map on R^2 by

$$f(x,y) = (y, g(y) + \sin[(y - g(x))(bg'(y) + c)]). \qquad (1)$$

We will be most interested in cases where g is a chaotic map. For example, $g(x) = a \sin \pi x$ with $a = 1.1$ has a chaotic at-

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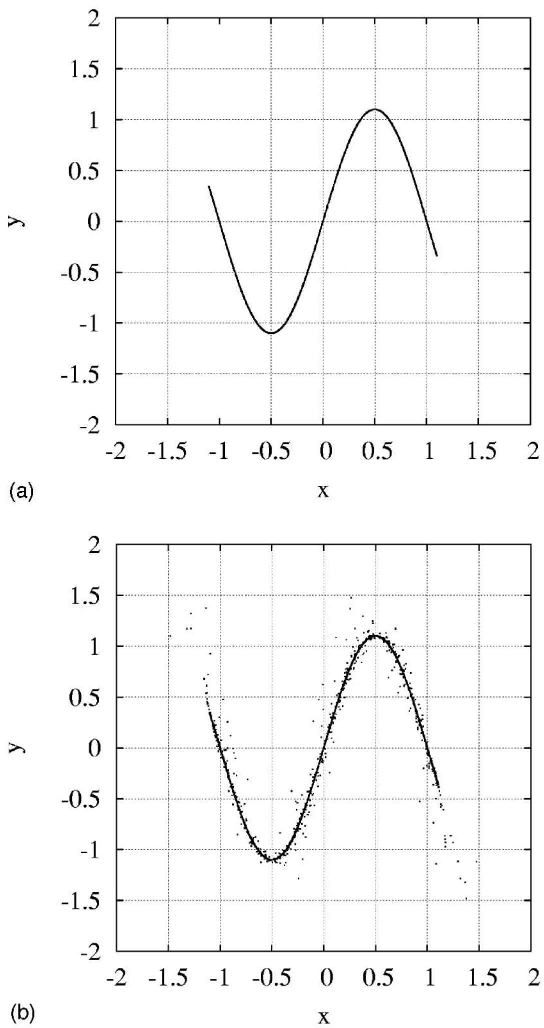

FIG. 1. The two-dimensional map (1) formed using the one-dimensional map $g(x)=1.1\sin \pi x$. (a) Invariant set under the deterministic system (2). (b) With added noise of size 10^{-16} per iteration. Four thousand iterations of a typical trajectory are shown in each frame.

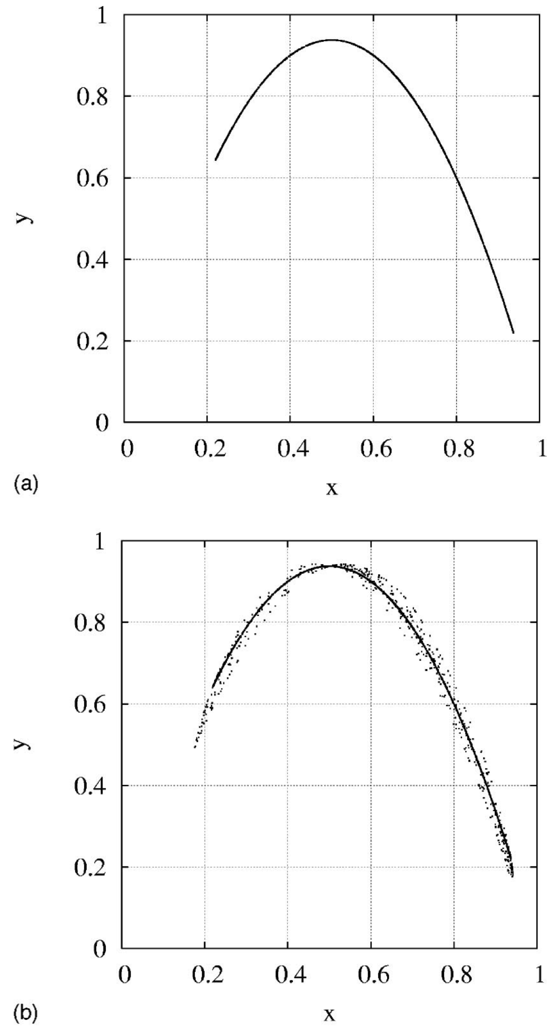

FIG. 2. The two-dimensional map (3) formed using the logistic map $g(x)=ax(1-x)$ with $a=3.75$. (a) Invariant set under the deterministic system (3). (b) With added noise of size 10^{-16} per iteration.

tractor contained in the interval $[-1.1,1.1]$, as shown in Fig. 1(a). Substituting in Eq. (1) results in the two-dimensional map

$$f(x,y)=(y,a\sin\pi y+\sin[(y-a\sin\pi x)\times(b\pi\cos\pi y+c)]). \tag{2}$$

With parameter settings $b=0.47$ and $c=0.1$, this map has some interesting properties. A trajectory started on the "ghost" of the original attractor, defined by $y=g(x)$ in the plane, will remain on that set. However, adding noise of size 10^{-16}, the typical noise present in standard double precision computations, results in typical trajectories as in Fig. 1(b). The tiny added noise has in fact changed the natural measure of the attractor. More precisely, the stochastic dynamical system consisting of the deterministic dynamics together with the noise[6] has a natural measure whose difference from the original natural measure is around 15 orders of magnitude larger than the noise level.

Another interesting example is

$$f(x,y)=(y,ay(1-y)+S((y-ax(1-x))\times(ba(1-2y)+c))), \tag{3}$$

where $g(x)=ax(1-x)$ is the logistic map, and where the sin function of Eq. (1) has been replaced by the sigmoid function $S(x)=\max(\min(x,d),-d)$ for $d>0$. Note that $S(x)$ is similar to the sin function for small x: $S(x)=x$ for $|x|\leq d$, and $S(x)=\mathrm{sign}(x)d$ for $|x|>d$. The purpose of $S(x)$ is to restrain the two dimensional dynamics from leaving the primary basin of the chaotic attractor. In each case we will choose parameter a so that the dynamics of g are chaotic, and choose parameters b, c, and d such that the dynamics of f are bounded and the effects of chaotic itinerancy are clearly visible.

For example, setting $b=0.7$, $c=0.0$, and $d=0.05$ in Eq. (3) leads to the dynamics shown in Fig. 2. Part (a) shows an invariant set corresponding to the original one-dimensional logistic map attractor, whose natural measure is extremely sensitive to added noise, as shown in Fig. 2(b). Although on first glance Fig. 2(b) looks simply like Fig. 2(a) with added noise of size 10^{-1}, something more complex is happening. It is a combination of noise of size 10^{-16} and itinerant dynamics that causes the strange effect.

The sensitivity of natural measure to small noise is a property of chaotic itinerancy. This property is related to so-called fluctuating Lyapunov exponents. In order to construct examples like this, the parameters b and c are tuned to make one of the Lyapunov exponents close to zero. In particular, we will manipulate the finite-time Lyapunov exponents. The time-t Lyapunov exponents of a chaotic trajectory are the averages λ_i of the logarithm of local expansion rates along the trajectory of length t, so that an infinitesimal sphere of radius dr at the beginning of the trajectory would evolve to an ellipsoid with axes $\lambda_i^t dr$ after t time units. An infinite-length trajectory possesses a distribution of time-t Lyapunov exponents for any fixed t. The sense in which we will consider a Lyapunov exponent "close to zero"[7] is that along the trajectory, the mean of the finite-time Lyapunov exponent is small compared to its variance. Figure 3 shows the probability distributions of the time-t Lyapunov exponents of map (2), for $t=100$.

The classical Lyapunov exponents are the limits of the means of the finite-time distributions in Fig. 3, as $t\rightarrow\infty$. The width of the distributions tends to zero for large t. In fact, because the time-t Lyapunov exponents are essentially averages of t log expansions, we should expect asymptotically that the standard deviation of the distributions should scale as $t^{-1/2}$. We will exploit this fact to estimate the standard deviation of the time-1 Lyapunov exponent distribution as $\sigma_1 \approx \sqrt{t}\sigma_t$ for large t. Using large t for this purpose breaks up the short-term correlations, giving a standard deviation that is more consistent with the long-time asymptotics, and more stable to calculate, than the standard deviation directly calculated from the distribution of time-1 exponents.

The characteristic wandering of trajectories that correspond to fluctuating Lyapunov exponents are the essence of chaotic itinerancy, and are closely related to the studies of riddled basins.[8] In both cases, as a parameter is varied, extremely complicated bifurcation sequences are caused, in particular bubbling and blowout bifurcations.[9] The maps developed in this section are relatively simple but interesting examples of these phenomena. In this article, we will focus primary attention on only one implication of these properties, the hypersensitivity of the natural measure to small additive noise.

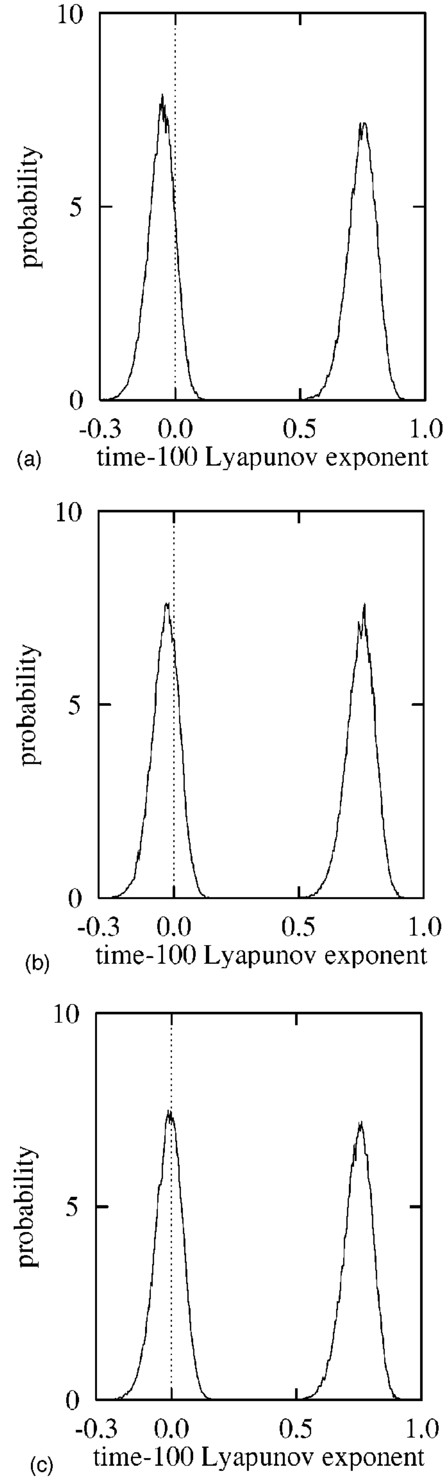

FIG. 3. Finite-time Lyapunov exponents of map (2) with $c=0.1$ and various settings of parameter b. Histograms of FTLE computed over nonoverlapping sections of 100 iterates are shown. (a) $b=0.45$, (b) $b=0.46$, (c) $b=0.47$.

III. STOCHASTIC DYNAMICAL SYSTEMS AND NATURAL MEASURE

The field of deterministic modeling of natural and experimental phenomena is in the process of absorbing the far-reaching effects of nonlinearity, including the possibility of chaos. In virtually all applications of deterministic modeling, parts of the process remain unmodeled, and are considered to be noise. The effectiveness of a deterministic model often depends on whether the so-called noise can be safely ignored or at least estimated.

In cases where the object of dynamical simulations is to compute a long-term average, the critical question is whether the effects of noise will alter the average. The question can be posed more generally in terms of natural measure, defined to be the invariant measure generated by almost every initial condition in the sense of probability. The question is whether the natural measure of the stochastic dynamical system formed by adding noise is close to the natural measure of the original deterministic system. Is the determination of the natural measure a well-conditioned problem?

The goal of this article is to suggest that systems with chaotic itinerancy give examples where the condition number of this problem is large. For a deterministic map $f(x,y)$, define the stochastic system with dynamical Gaussian noise by

$$f_\delta(x,y)=f(x,y)+\delta\eta, \tag{4}$$

where δ is a fixed noise level and $\eta\in N(0,1)$ is chosen from the standard normal distribution. (The shape of the noise distribution is not critical to the results shown in the following. Results obtained with uniform noise of similar strength differ only slightly.)

Figure 4 illustrates the change in natural measure due to the addition of additive Gaussian noise of size $\delta=10^{-12}$. Each part of Fig. 4 shows 10^7 iterations of map (4) where f is defined in Eq. (2). The attractor from Fig. 1 is still visible (note that only a small part of the original area is shown) in part (a), corresponding to parameter $b=0.45$. The chaotic itinerancy becomes more noticeable in Fig. 4(b), where the parameter $b=0.46$. Finally, when $b=0.47$ as in (c), the natural measure has undergone a significant change.

IV. SCALING LAW FOR NATURAL MEASURE

What causes one-step errors at the fine scale of 10^{-12} to be manifested in unit-sized changes in the attractor? It is helpful to compare the extreme changes in measure with the fluctuating Lyapunov exponents seen in Fig. 3. As the parameter b moves from 0.45 to 0.47, the finite-time Lyapunov exponent closest to zero moves ever closer to zero, in the sense that the mean of the distribution becomes small compared to the variance.

In an article on shadowing of numerical trajectories,[3] a scaling law was developed to quantify the size of excursions from deterministic trajectories caused by additive noise. The scaling law says that the distribution of the log distances along the trajectory between the noise-free and noisy trajectories is approximately an exponential distribution

$$p(y)=he^{-h(y-\ln\delta)} \tag{5}$$

for $y\geqslant\ln\delta$, where h is a scaling exponent quantifying the fluctuation of Lyapunov exponents:

$$h=\frac{2|m|}{\sigma^2}, \tag{6}$$

called the *hyperbolicity exponent*. Here m and σ represent the mean and standard deviation, respectively, of the distribution of the finite-time Lyapunov exponent closest to zero, scaled to time $t=1$. The exponential distribution is the result of tiny excursions that periodically move the noisy trajectory away from the original trajectory, and then return toward it.

This scaling law was applied in Ref. 10 to develop a scaling law for the change in natural measure, or more generally in any averaged observation function on the state space, due to additive noise. Recall that f_δ denotes the noisy version of f, defined in Eq. (4). It was conjectured in Ref. 10 that for a fixed observation function $r(x)$ defined on the state space of the system, the difference between the noisy trajectory average of r under f_δ and the original trajectory average of r under f scales with the one-step error δ with a scaling

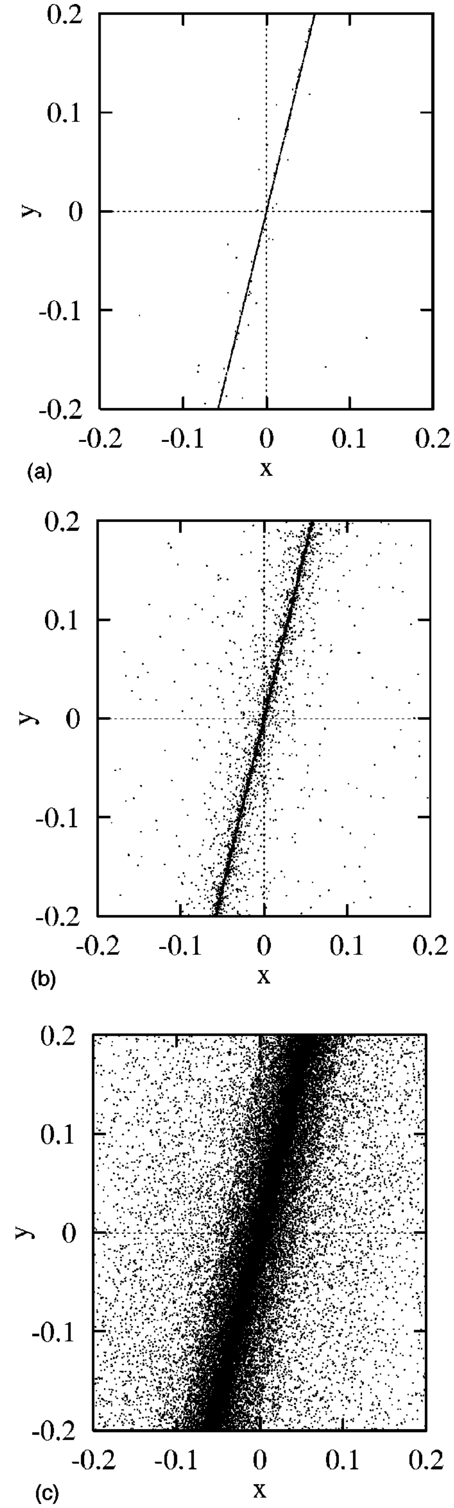

FIG. 4. Natural measure of Eq. (4) with $\delta=10^{-12}$. The points from a length-10^7 trajectory that fall into the square $-0.2\leqslant x,y\leqslant 0.2$ are plotted. (a) $b=0.45$, (b) $b=0.46$, (c) $b=0.47$.

exponent of $h=2m/\sigma^2$, or in terms of expectation over the natural measure of f_δ and the natural measure of f:

$$\langle r\rangle_{f_\delta}-\langle r\rangle_f=K\delta^h, \tag{7}$$

where h represents the hyperbolicity exponent measured from the Lyapunov exponent lying closest to zero. The scaling formula is an asymptotic formula, which holds in the

TABLE I. Finite-time Lyapunov exponent statistics for two-dimensional map (1). Only the second-largest Lyapunov exponent is considered here, because it is the one that fluctuates around zero. The mean and standard deviation of the time-100 Lyapunov exponent are denoted by m and σ_{100}, respectively. The hyperbolicity exponent h is computed from Eq. (6).

b	c	m	σ_{100}	σ_1	h
0.45	0.1	−0.0557	0.0545	0.545	0.38
0.46	0.1	−0.0335	0.0545	0.545	0.22
0.47	0.1	−0.0122	0.0548	0.548	0.08

limit as $\delta \to 0$, and to the extent that one Lyapunov exponent is significantly closer to zero than all others. The situation will be more complicated when these assumptions are violated.

Equation (7) clarifies the role of a fluctuating Lyapunov exponent in the sensitivity of natural measure to small noise perturbations. If the hyperbolicity exponent h is near 1, little propagation of error will occur from the small scale to the large. Additive Gaussian noise of size δ will cause "errors" in the natural measure of about the same scale δ. On the other hand, if $h \approx 0$, there may be a large difference in expected values, of the observation function $r(x)$ under the noisy and deterministic probability measures. Note that the proportionality constant K in Eq. (7) is dependent on the observation function r. In fact, K may be zero or small enough to make the bias undetectably small for some choices of r, and large for others.

We can compute the hyperbolicity exponent h from the moments of the lower finite-time Lyapunov exponent shown in Fig. 3. They are given in Table I. For parameter $b = 0.45$, the mean and standard deviation of the time-100 Lyapunov exponents are -0.0557 and 0.0545, respectively. To scale the standard deviation from time 100 to time 1 requires multiplying by $\sqrt{100} = 10$. The hyperbolicity exponent is computed as

$$h = \frac{2|m|}{\sigma_1^2} = \frac{2(0.0557)}{(0.545)^2} \approx 0.38$$

from formula (6).

The choice of time $t = 100$ is arbitrary. As mentioned earlier, the calculation of the standard deviation of the finite-time Lyapunov exponents distribution is simplified when $t \gg 1$ is used. Any large t would work as well, and give an approximately similar value of h.

For the nearby parameter setting $b = 0.47$, the result is completely different. The hyperbolicity exponent is $h \approx 0.08$, meaning that reduction in additive noise δ of more than 10 orders of magnitude is required to reduce error in natural measure by a single order of magnitude. In this case natural measure is extremely sensitive to additive noise.

Figure 5 illustrates the bias in long-term averages caused by additive noise, as predicted by formula (7). Two observation functions r, designed to register maximum impact of the change in measure evident in Fig. 4, were selected. The upper curve in Fig. 5 plots the average over 10^7 iterates of $r(x,y) = (y - g(x))^2$. This directly measures the deviation of the noisy attractor from the original one-dimensional deterministic attractor, which falls on the curve $y = g(x)$. The lower curve plots the average of the indicator function on the square Q defined by $0 \leqslant x \leqslant 0.2, -0.2 \leqslant y \leqslant 0$, the lower-right quarter of Fig. 4. The lower curve therefore directly calculates the natural measure on a fixed square in state space. The slopes of the best line fit to the averages are approximately 0.4,0.24, and 0.09, respectively, for $b = 0.45, 0.46, 0.47$.

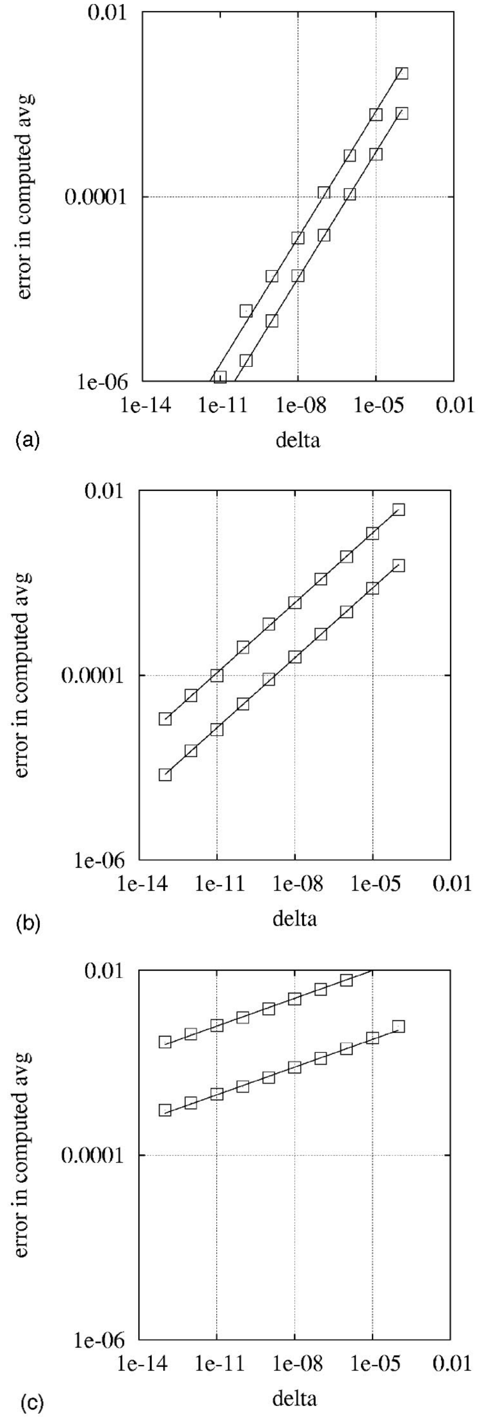

FIG. 5. Changes, by adding δ noise, in long-term averages of two observation functions under the dynamics of Eq. (1). The functions corresponding to the top, respectively bottom curve are $r(x,y) = (y - g(x))^2$ and $r(x,y) = 1_Q$, where Q is the set $[0,0.2] \times [-0.2,0]$. (a) $b = 0.45$, (b) $b = 0.46$, (c) $b = 0.47$.

These compare closely with the hyperbolicity exponents in Table I, as expected by Eq. (7).

V. DISCUSSION

We have presented a framework for constructing examples of chaotic itinerancy through one-dimensional maps, but by slightly modifying Eq. (1) it can be used with higher-dimensional attractors as well. The result will be a deterministic attractor whose natural measure is hypersensitive to added noise for certain values of the parameters.

Considerations about natural measure are critical in most dynamical simulations where averages are sought. Numerical shadowing studies[11] are concerned about this problem, when the noise comes from machine rounding or discretization error. Although these forms of noise are not Gaussian, it is likely that the qualitative issues are the same as in the Gaussian case.

Whether a long-term average computed from a simulation is susceptible to noise depends also on the observation function that is being averaged. If the constant K in Eq. (7) is small, the average may still be computed accurately even when the hyperbolicity exponent h is near zero.

The key formula (7) for change in natural measure is an asymptotic limit for the case when one finite-time Lyapunov exponent distribution is nearer to zero than the others. It would be interesting to develop a more sophisticated formula that gives more information about the general case and more accuracy in all contexts.

VI. ACKNOWLEDGMENT

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[1] K. Kaneko, Physica D **41**, 137 (1990).
[2] I. Tsuda, World Futures **32**, 167 (1990).
[3] T. Sauer, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **79**, 59 (1997).
[4] C. Schroer, E. Ott, and J. A. Yorke, Phys. Rev. Lett. **81**, 1397 (1998).
[5] V. Anishchenko, T. Vadivasova, A. Kopeikin, J. Kurths, and G. Strelkova, Phys. Rev. Lett. **87**, 054101 (2001).
[6] L. Arnold, *Random Dynamical Systems* (Springer, Berlin, 1998).
[7] S. Dawson, C. Grebogi, T. Sauer, and J. A. Yorke. Phys. Rev. Lett. **73**, 1927 (1994).
[8] J. C. Alexander, J. A. Yorke, Z. You, and I. Kan, Int. J. Bifurcation Chaos Appl. Sci. Eng. **2**, 795 (1992).
[9] E. Ott and J. C. Sommerer, Phys. Lett. A **188**, 38 (1994); P. Ashwin, J. Buescu, and I. N. Stewart, *ibid.* **193**, 126 (1994); S. Venkataramani, B. Hunt, and E. Ott, Phys. Rev. E **54**, 1346 (1996).
[10] T. Sauer, Phys. Rev. E **65**, 036220 (2002).
[11] C. Grebogi, S. Hammel, and J. Yorke, Bull. Am. Math. Soc. **19**, 465 (1988); C. Grebogi, S. Hammel, J. A. Yorke, and T. Sauer, Phys. Rev. Lett. **65**, 1527 (1990); S-N. Chow and K. Palmer, J. Dyn. Diff. Eqns. **3**, 361 (1991); J. M. Sanz-Serna and S. Larsson, Appl. Numer. Math. **13**, 181 (1993); W. Hayes, Phys. Rev. Lett. **90**, 054104 (2003).