Convergence of Rank-Type Equations

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Abstract

Convergence results are presented for rank-type difference equations, whose evolution rule is defined at each step as the kth largest of p univariate difference equations. If the univariate equations are individually contractive, then the equation converges to a fixed point equal to the kth largest of the individual fixed points of the univariate equations. Examples are max-type equations for $k = 1$, and the median of an odd number p of equations, for $k = (p+1)/2$. In the non-hyperbolic case, conjectures are stated about the eventual periodicity of the equations, generalizing long-standing conjectures of G. Ladas.

1 Introduction

For a set of p real numbers $\{r_1, \ldots, r_p\}$, denote the kth-largest element of the set by $k\text{-rank}\{r_1, \ldots, r_p\}$. Thus $2\text{-rank}\{6, 2, 5, 3\} = 5$, and 1-rank is synonymous with max.

Let $f_i : R \to R$ for $i = 1, \ldots, p$ be real-valued functions. Consider the difference equation

$$x_n = k\text{-rank}\{f_1(x_{n-1}), f_2(x_{n-2}), \ldots, f_p(x_{n-p})\} \tag{1}$$

for initial data $x_1, \ldots, x_p$. We will call such an equation a *rank-type* difference equation. If the f_i are continuous, then x_n is a continuous function of $x_{n-1}, \ldots, x_{n-p}$. Special cases of rank-type equations include

$$x_n = \max\{f_1(x_{n-1}), f_2(x_{n-2}), \ldots, f_p(x_{n-p})\}, \tag{2}$$

$$x_n = \min\{f_1(x_{n-1}), f_2(x_{n-2}), \ldots, f_p(x_{n-p})\}, \tag{3}$$

and

$$x_n = \text{median}\{f_1(x_{n-1}), f_2(x_{n-2}), \ldots, f_p(x_{n-p})\}. \tag{4}$$

in the case where p is odd.

Max-type equations, corresponding to the special case $k = 1$ in difference equation (1), have been extensively studied [1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18]. The purpose of this article is to note that, perhaps surprisingly, many of the properties of max-type equations are shared more generally by rank-type equations for $k > 1$.

Definition The function f is called *contractive* if there exists $0 \leq \alpha < 1$ and a real number r such that $|f(x) - r| \leq \alpha|x - r|$ for all x.

Definition The solution $\{x_n\}_{n=1}^{\infty}$ of a difference equation is called *globally convergent* if there exists r such that for every set of initial values, $\lim_{n\to\infty} x_n = r$. In this case, the equilibrium r is called *globally attractive.*

In the next section we show that if the f_i are contractive with fixed points r_i, then the difference equation (1) is globally convergent for any set $\{x_1, \dots, x_p\}$ of initial values, converging in the limit to the fixed point k-rank$\{r_1, \dots, r_p\}$.

This result is a generalization of the convergence theorem for max-type equations, the $k = 1$ case [11]. If p is odd and $k = (p + 1)/2$, then the convergence result corresponds to replacing max by median. The corresponding statement for mean is false; see Example 2.6.

Theorem 2.3 below is the main global convergence result, proved in a context slightly more general than (1). The techniques used to prove Theorem 2.3 can also be applied to prove a local convergence version, Theorem 3.1. In the final section, we relax the hyperbolicity restriction and state some conjectures, generalizing well-known conjectures of Ladas [9, 7] on max-type equations.

2 Global convergence

The following lemma from [11] is required.

Lemma 2.1 *Let p be a positive integer, r and $0 \leq \alpha < 1$ real numbers, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Assume that for each n there exists i, possibly depending on n, $1 \leq i \leq p$, such that $|x_n - r| \leq \alpha|x_{n-i} - r|$. Then* $\lim_{n\to\infty} x_n = r$.

The next lemma generalizes Lemma 2.2 of [11].

Lemma 2.2 *Let* $u_1, u_2,\ y_1 \le y_2,$ *and* $s_2 \le r \le s_1$ *be real numbers, and assume* $|y_i - s_i| \le \alpha |u_i - s_i|$ *for* $i = 1, 2$ *for some* $0 \le \alpha < 1$. *Then*

(i) $|y_2 - r| \le \alpha |u_j - r|$ *for either* $j = 1$ *or* $j = 2$, *and*

(ii) $|y_1 - r| \le \alpha |u_j - r|$ *for either* $j = 1$ *or* $j = 2$.

Proof We give the proof of *(i)*. The proof of *(ii)* follows by applying *(i)* to $-u_1, -u_2, -y_2, -y_1, -s_1, -r, -s_2$.

The proof of *(i)* is divided into four cases.

Case 1: $y_2 \le r, u_1 \ge s_1$. In this case,

$$\begin{aligned} |y_2 - r| &= r - y_2 \le s_1 - y_2 \le s_1 - y_1 = |s_1 - y_1| \\ &\le \alpha |s_1 - u_1| = \alpha(s_1 - u_1) \le \alpha(u_1 - r) = \alpha |u_1 - r|. \end{aligned}$$

Case 2: $y_2 \le r, u_1 \le s_1$.

$$\begin{aligned} |y_2 - r| &= r - y_2 \le r - y_1 = r - s_1 + s_1 - y_1 = r - s_1 + |y_1 - s_1| \\ &\le r - s_1 + \alpha |u_1 - s_1| = r - s_1 + \alpha(s_1 - u_1) \\ &\le r - s_1 + \alpha(s_1 - u_1) + (1 - \alpha)(s_1 - r) = \alpha(r - u_1) \le \alpha |r - u_1|. \end{aligned}$$

Case 3: $y_2 \ge r, u_2 \ge r$.

$$\begin{aligned} |y_2 - r| &= y_2 - r = y_2 - s_2 + s_2 - r = |y_2 - s_2| + s_2 - r \\ &\le \alpha |u_2 - s_2| + s_2 - r = \alpha(u_2 - s_2) + s_2 - r \\ &\le \alpha(u_2 - s_2) + s_2 - r + (1 - \alpha)(r - s_2) = \alpha(u_2 - r) = \alpha |u_2 - r|. \end{aligned}$$

Case 4: $y_2 \ge r, u_2 \le r$.

If $u_2 \ge s_2$, then

$$r - s_2 \le y_2 - s_2 = |y_2 - s_2| \le \alpha |u_2 - s_2| = \alpha(u_2 - s_2) \le \alpha(r - s_2),$$

a contradiction. So in addition, we may assume $u_2 < s_2 \le r$. Then

$$\begin{aligned} |y_2 - r| &= y_2 - r \le y_2 - s_2 = |y_2 - s_2| \\ &\le \alpha |u_2 - s_2| = \alpha(s_2 - u_2) \le \alpha(r - u_2) = \alpha |u_2 - r|, \end{aligned}$$

which completes the proof.

Theorem 2.3 *Consider* p *nonnegative integers* $q_1, \ldots, q_p$, *and let* $0 \le \alpha < 1$. *Assume for each* i, j *satisfying* $1 \le i \le p, 1 \le j \le q_i$ *there exists a function* $f_{ij} : R \to R$ *and a real number* r_{ij} *satisfying*

$$|f_{ij}(x) - r_{ij}| \le \alpha |x - r_{ij}|$$

for all x. Then for any k and for any set $\{x_1, \ldots, x_p\}$ of initial values, the solution of the difference equation

$$x_n = k\text{-}rank_{1\leq i\leq p, 1\leq j\leq q_i}\{f_{ij}(x_{n-i})\} \tag{5}$$

converges to k-rank$_{1\leq i\leq p,1\leq j\leq q_i} r_{ij}$, the kth-largest of the r_{ij}.

Proof There are $q \equiv \sum_{i=1}^{p} q_j$ functions f_{ij}, each with fixed point r_{ij}. Rank the q fixed points as follows:

$$r_{i_1 j_1} \geq r_{i_2 j_2} \geq \cdots \geq r_{i_q j_q}.$$

We need to prove that the sequence $x_n = k\text{-rank}\{f_{ij}(x_{n-i})\}$ converges to $r \equiv r_{i_k j_k} = k\text{-rank } r_{ij}$.

To apply Lemma 2.1, we need to find $|x_n - r| \leq \alpha|x_{n-i} - r|$ for some $1 \leq i \leq n$. For a fixed n, define i_m, j_m so that $x_n \equiv k\text{-rank}_{i,j}\{f_{ij}(x_{n-i})\} = f_{i_m j_m}(x_{n-i_m})$. To apply Lemma 2.1, we will find x_{n-i} satisfying $|x_n - r| \leq \alpha|x_{n-i} - r|$ where $1 \leq i \leq p$.

If $m = k$, then $|x_n - r| \leq \alpha|x_{n-i_k} - r|$, as required. If $m < k$, then there exists an integer $h \in \{k, k+1, \ldots, q\}$ such that $f_{i_h j_h}(x_{n-i_h}) \geq f_{i_m j_m}(x_{n-i_m})$. Thus $m < k \leq h$, or in other words, $r_{i_m j_m} \geq r_{i_k j_k} \geq r_{i_h j_h}$. Now we can apply Lemma 2.2 with $s_2 = r_{i_h j_h} \leq r = r_{i_k j_k} \leq s_1 = r_{i_m j_m}, y_1 = f_{i_m j_m}(x_{n-i_m}) \leq y_2 = f_{i_h j_h}(x_{n-i_h})$, and $u_1 = x_{n-i_m}, u_2 = x_{n-i_h}$. The result of part (ii) of the lemma is that

$$|x_n - r| = |k\text{-rank}_{1\leq i\leq p,1\leq j\leq q_i}\{f_{ij}(x_{n-i})\} - r| \leq \alpha|z - r| \tag{6}$$

where $z = x_{n-i_m}$ or x_{n-i_h}, as required. Finally, if $m > k$, there exists $h \in \{1, 2, \ldots, k\}$ such that $f_{i_h j_h}(x_{n-i_h}) \leq f_{i_m j_m}(x_{n-i_m})$. Thus $h \leq k < m$, or in other words, $r_{i_h j_h} \geq r_{i_k j_k} \geq r_{i_m j_m}$. Part (i) of Lemma 2.2 with $s_2 = r_{i_m j_m} \leq r = r_{i_k j_k} \leq s_1 = r_{i_h j_h}, y_1 = f_{i_h j_h}(x_{n-i_h}) \leq y_2 = f_{i_m j_m}(x_{n-i_m})$, and $u_1 = x_{n-i_h}, u_2 = x_{n-i_m}$ yields (6) as before.

This satisfies the hypotheses of Lemma 2.1, so

$$\lim_{n\to\infty} x_n = r = k\text{-rank } \{r_{ij}\}.$$

Setting all $q_i = 1$ in Theorem 2.3 covers the special case referred to as equation (1) in the introduction:

Corollary 2.4 *Let $r_1, \ldots, r_p$ be real numbers and assume $f_i : R \to R$ for $i = 1, \ldots, p$ satisfy $|f_i(x) - r_i| \leq \alpha|x - r_i|$ for all x, where $0 \leq \alpha < 1$. Then for any set $\{x_1, \ldots, x_p\}$ of initial values, the solution of difference equation*

$$x_n = k\text{-}rank\ \{f_1(x_{n-1}), \ldots, f_p(x_{n-p})\} \tag{7}$$

converges to k-rank$\{r_1, \ldots, r_p\}$ as $n \to \infty$.

Example 2.5 As an application of Corollary 2.4, consider the difference equation

$$x_n = k\text{-rank}\{\frac{1}{a_1 + b_1 x_{n-1}^2}, \ldots, \frac{1}{a_p + b_p x_{n-p}^2}\} \tag{8}$$

where $0 < a_i, 0 \leq b_i < (a_i + \frac{1}{4})^4$ for $i = 1, \ldots, p$. Under these conditions, for each i, the first derivative of $f_i(x) = 1/(a_i + b_i x^2)$ is smaller than 1 in absolute value for all x. By the Mean Value Theorem, the hypotheses of Corollary 2.4 hold where r_i denotes the real root of the equation $b_i x^3 + a_i x = 1$. Therefore all solutions of (8) must converge to k-rank$\{r_1, \ldots, r_p\}$.

A particular case of (8) for $k = 2$ is the difference equation

$$x_n = \text{median}\left\{\frac{1}{1.2 + 0.7x_{n-1}^2}, \frac{1}{1 + x_{n-2}^2}, \frac{1}{1.1 + 0.9x_{n-3}^2}\right\} \tag{9}$$

The fixed points of

$$f_1(x) = \frac{1}{1.2 + 0.7x^2}, \quad f_2(x) = \frac{1}{1 + x^2}, \quad f_3(x) = \frac{1}{1.1 + 0.9x^2}$$

are approximately $r_1 = 0.6632, r_2 = 0.6823$, and exactly $r_3 = 2/3$, respectively. Corollary 2.4 implies that all solutions of (9) converge to $r_3 = 2/3$, the median of the three fixed points.

Remark Corollary 2.4 implies that if p is odd, then the median difference equation (4) converges to the median of the individual fixed points of $f_1, \ldots, f_p$. However, if p is even, this statement fails to hold, as shown in the next example.

Example 2.6 If $a_1 + a_2 < 2$ for positive numbers a_1, a_2, the equation

$$x_n = \text{median}\{a_1 x_{n-1} + b_1, a_2 x_{n-2} + b_2\} \tag{10}$$

is equivalent to

$$x_n = \text{mean}\{a_1 x_{n-1} + b_1, a_2 x_{n-2} + b_2\}$$

and converges to the fixed point

$$\lim_{n\to\infty} x_n = (b_1 + b_2)/(2 - a_1 - a_2).$$

This disagrees in general with the mean of the fixed points of $f_1(x) = a_1 x + b_1$ and $f_2(x) = a_2 x + b_2$, which is $b_1/2(1 - a_1) + b_2/2(1 - a_2)$.

Example 2.7 Consider the difference equation

$$x_n = k\text{-rank}\{A_1 x_{n-1}^{\alpha_1}, \dots, A_p x_{n-p}^{\alpha_p}\} \tag{11}$$

where $A_i > 0, -1 < \alpha_i < 1$ for $i = 1, \dots, p$, and $x_1, \dots, x_p$ are initial values. Set $y_n = \log x_n$. In these coordinates, the ith equation is $y_n = \alpha_i y_{n-i} + \log A_i$, and due to monotonicity of the logarithm, (11) is replaced with

$$y_n = k\text{-rank}\{\alpha_1 y_{n-1} + \log A_1, \dots, \alpha_p y_{n-p} + \log A_p\}.$$

Corollary 2.4 shows that for any set of positive initial values $x_1, \dots, x_p$, the y_n sequence converges to k-rank$\{\log A_i/(1-\alpha_i)\}$, so that

$$\lim_{n\to\infty} x_n = k\text{-rank}_{1\le i\le p} A_i^{\frac{1}{1-\alpha_i}}.$$

This proves asymptotic convergence of (11) for $-1 < \alpha_i < 1, A_i > 0$, and for all positive initial conditions. The max-type version of the problem, corresponding to $k = 1$, was previously treated in [14, 12, 11].

3 Local convergence

Definition The constant solution $x_n = r$ of a difference equation wiil be called *locally attractive* if for some p-dimensional open neighborhood of initial values $(x_1, \dots, x_p) = (r, \dots, r)$, the solution converges to the constant solution r.

This definition concerns local convergence, for cases when nearby initial values are attracted to a given constant solution. In the context of rank-type equations, in order to make conclusions about local convergence, an extra hypothesis that is not strictly local needs to be added to control the contractivity between the individual fixed points, as shown in the next theorem.

Theorem 3.1 *Consider p nonnegative integers $q_1, \dots, q_p$, and let $0 \le \alpha < 1$. Assume for each i, j satisfying $1 \le i \le p, 1 \le j \le q_i$ there exists a continuously differentiable function $f_{ij} : R \to R$ and a real number r_{ij} satisfying $f_{ij}(r_{ij}) = r_{ij}$. Let i_k, j_k be integers satisfying $r_{i_k j_k} = k\text{-rank}_{i,j} r_{ij}$. Assume that for each $i, j, |f'_{ij}(x)| \le \alpha$ for x between r_{ij} and $r_{i_k j_k}$. Then the constant solution $x_n = r_{i_k j_k}$ of the rank-type difference equation*

$$x_n = k\text{-}rank_{1\le i\le p, 1\le j\le q_i}\{f_{ij}(x_{n-i})\} \tag{12}$$

is locally attractive.

Proof Choose $\epsilon > 0$ such that for each i, j, $|f'_{ij}(x)| \leq \alpha_1 \equiv (\alpha+1)/2 < 1$ for $r_{ij} - \epsilon < x < r_{i_k j_k} + \epsilon$. For each i, j and $r_{ij} - \epsilon < x < r_{i_k j_k} + \epsilon$, the Mean Value Theorem implies $|f_{ij}(x) - r_{ij}| \leq \alpha_1 |x - r_{ij}|$. Define the open set $U = \{(x_1, \ldots, x_p) : |x_i - r_{i_k j_k}| < \epsilon, 1 \leq i \leq p\}$.

The remainder of the proof is similar to the proof of Theorem 2.3. Choose $(x_1, \ldots, x_p)$ from U, and for each $n > p$, choose i', j' such that $x_n = \max_{i,j}\{f_{ij}(x_{n-i})\} = f_{i'j'}(x_{n-i'})$. Apply Lemma 2.2 with $u_1 = x_{n-i_k}, y_1 = f_{i_k j_k}(x_{n-i_k}), u_2 = x_{n-i'}, y_2 = f_{i'j'}(x_{n-i'}), s_1 = r_{i_k j_k}$, and $s_2 = r_{i'j'}$. Lemma 2.2 implies that

$$|x_n - r_{i_k j_k}| = |\max_{i,j}\{f_{ij}(x_{n-i})\} - r_{i_k j_k}| \leq \alpha_1 |z - r_{i_m j_m}|$$

where $z = x_{n-i_k}$ or $x_{n-i'}$. This implies that (a) x_n belongs to U and (b) we can apply Lemma 2.1 to conclude that $\lim_{n\to\infty} x_n = r_{i_k j_k}$.

The $q_i \equiv 1$ special case is the local version of Corollary 2.4.

Corollary 3.2 *Assume that the continuously differentiable functions $f_i : R \to R$ and real numbers r_i for $i = 1, \ldots, p$ satisfy $f_i(r_i) = r_i$. Let i_k be an integer satisfying $r_{i_k} = k\text{-rank}_{1\leq i\leq p} r_i$, and assume that there exists $0 \leq \alpha < 1$ such that for $1 \leq i \leq p$, $|f'_i(x)| \leq \alpha$ for x between r_i and r_{i_k}. Then the constant solution $x_n = r_{i_k}$ of the difference equation*

$$x_n = k\text{-}rank\{f_1(x_{n-1}), \ldots, f_p(x_{n-p})\} \tag{13}$$

is locally attractive.

We revisit two examples of max-type equations from [11], and discuss them in the more general context of Corollary 3.2.

Example 3.3 As a first example, consider the rank-type equation involving Ricker maps [10]

$$x_n = k\text{-rank}\{x_{n-1}e^{a_1(1-x_{n-1}/c_1)}, \ldots, x_{n-p}e^{a_p(1-x_{n-p}/c_p)}\} \tag{14}$$

where each map $f_i(x) = xe^{a_i(1-x/c_i)}$ in (13) has growth parameter $a_i \geq 0$ and carrying capacity $c_i \geq 0$. Since $f'_i(0) = e^{a_i} \geq 1$, f_i is not contractive, and the hypotheses of Corollary 2.4 are not satisfied.

However, note that If $0 < a_i < 2$, then c_i is a stable fixed point for f_i, since the derivative of $f_i(x) = xe^{a_i(1-x/c_i)}$ is $f'_i(x) = (1 - a_i x/c_i)e^{a_i(1-x/c_i)}$, and $|f'_i(c_i)| = |1-a_i| < 1$. In addition, the second derivative shows that $f'_i(x)$

is decreasing on the interval $[c_i, 2c_i/a_i)$ from $f_i'(c_i) = 1 - a_i$ to $f_i'(2c_i/a_i) = -e^{a_i-2}$, and increasing on the interval $(2c_i/a_i, \infty)$ from $f_i'(2c_i/a_i) = -e^{a_i-2}$ to 0. It follows immediately that $|f_i'(x)| \leq \max\{|1 - a_i|, e^{a_i-2}\} < 1$ for $c_i \leq x$. Now the main hypothesis of Corollary 3.2, that for each i, $|f_i'(x)| = |(1 - a_i x/c_i)e^{a_i(1-x/c_i)}| \leq \alpha \equiv \max_i\{|1 - a_i|, e^{a_i-2}\} < 1$ for x between c_i and c_{i_k}, is verified. Therefore the constant solution $\{c_{i_k}, c_{i_k}, \ldots\}$ is locally attractive for the rank-type equation (14), where $c_{i_k} = k\text{-rank}\{c_i\}$ is the kth-largest of the carrying capacities of the p individual Ricker maps.

Example 3.4 Assume $-1/4 < a_i < 3/4$ for $1 \leq i \leq p$. Then the fixed point $r_i = a_i + \frac{1}{2} - \sqrt{a_i + \frac{1}{4}}$ of $f_i(x) = (x - a_i)^2$ is an attracting fixed point. Note that each fixed point lies in the interval $[0, \frac{1}{4})$.

We can apply Corollary 3.2 to the difference equation

$$x_n = k\text{-rank}\{(x_{n-1} - a_1)^2, (x_{n-2} - a_2)^2, \ldots, (x_{n-p} - a_p)^2\} \qquad (15)$$

Note that for each j and for x between $x = r_i$ and $x = \frac{1}{4}$, $f_i'(x)$ is increasing from $f_i'(r_i) = 1 - \sqrt{4a_i + 1}$ to $f_i'(\frac{1}{4}) = 2(\frac{1}{4} - a_i)$, so that $|f_i'(x)| \leq \max\{|1 - \sqrt{4a_i + 1}|, |2(\frac{1}{4} - a_i)|\} < 1$, satisfying the main hypothesis of Corollary 3.2. It follows that the constant solution $x_n = r_{i_k}$, the kth-largest of the p individually attracting fixed points of the f_i, is locally attractive for the rank-type equation (15).

4 Non-hyperbolic case

Far less is known in the non-hyperbolic case, where the restriction that $\alpha < 1$ is relaxed. Because of the lack of hyperbolicity, convergence cannot be expected for general initial conditions. In many cases the solution becomes periodic for sufficiently large n. A particularly rich case concerns the difference equation

$$x_n = k\text{-rank}\{-x_{n-1} + b_1, \ldots, -x_{n-p} + b_p\}. \qquad (16)$$

Remark First define i_k to be the index of the kth-largest b_i, namely $b_{i_k} = k$-rank $\{b_1, \ldots, b_p\}$. Then it is straightforward to check that (16) has a fixed-point solution $x_1 = x_2 = \ldots = \frac{1}{2}b_{i_k}$.

With more assumptions, many more periodic solutions can be generated. We say that a solution has *prime period s* if it is periodic with period s and with no lower period.

Proposition 4.1 *Assume that the $b_1, \dots, b_p$ in (16) are ordered as*

$$b_{i_1} \geq \dots \geq b_{i_{k-1}} > b_{i_k} > b_{i_{k+1}} \geq \dots \geq b_{i_p}.$$

That is, assume that b_{i_k}, the kth-largest b_i, is not repeated in the list. Then there are uncountably many solutions of (16) with prime period $2i_k$.

The solutions can be described as follows. Define

$$\beta = \frac{1}{2}\min\{b_{i_{k-1}} - b_{i_k}, b_{i_k} - b_{i_{k+1}}\}.$$

(If $k = 1$, set $\beta = \frac{1}{2}(b_{i_k} - b_{i_{k+1}})$; if $k = p$, set $\beta = \frac{1}{2}(b_{i_{k-1}} - b_{i_k})$). Define the p initial conditions $x_1, \dots, x_p$ to be any p consecutive elements of the sequence

$$x_1, \dots, x_{2i_k}, x_1, \dots, x_{2i_k}, \dots$$

where

$$\begin{aligned}
x_1 &= \frac{b_{i_k}}{2} + \beta_1 \\
&\vdots \\
x_{i_k} &= \frac{b_{i_k}}{2} + \beta_{i_k} \\
x_{i_k+1} &= \frac{b_{i_k}}{2} - \beta_1 \\
&\vdots \\
x_{2i_k} &= \frac{b_{i_k}}{2} - \beta_{i_k}
\end{aligned}$$

and such that the β_i satisfy $|\beta_i| < \beta$. For each n,

$$x_n = k\text{-rank}\left\{-\frac{b_{i_k}}{2} - \beta_{j_1} + b_1, \dots, -\frac{b_{i_k}}{2} - \beta_{j_p} + b_p\right\}.$$

Since $b_{i_{k-1}} - 2\beta \geq b_{i_k} \geq b_{i_{k+1}} + 2\beta$, we have

$$-\frac{b_{i_k}}{2} + b_{i_{k-1}} - 2\beta \geq -\frac{b_{i_k}}{2} + b_{i_k} \geq -\frac{b_{i_k}}{2} + b_{i_{k+1}} + 2\beta,$$

so that the kth largest of the set is $-x_{n-i_k} + b_{i_k}$. This verifies that each x_n defined by the difference equation (16) follows the same pattern, and satisfies $x_n = x_{n-2i_k}$. If the $\beta_1, \dots, \beta_k$ are chosen all distinct, then the solution is not periodic of any lower period, so its prime period is $2i_k$.

Remark In addition to the solutions of prime period $2i_k$, there are solutions of prime period $2i_k/d$ for any odd natural number d dividing evenly into i_k. They are special cases of the above solutions obtained by setting $\beta_{s+1} = -\beta_1, \beta_{s+2} = -\beta_2, \ldots, \beta_{2s} = -\beta_s, \beta_{2s+1} = \beta_1, \ldots$, where $s = i_k/d$.

Note that difference equation (16) is the additive version of the multiplicative difference equation

$$x_n = k\text{-rank}\left\{\frac{A_1}{x_1}, \ldots, \frac{A_p}{x_p}\right\} \tag{17}$$

where $A_i > 0$. If we set $y_i = \log x_i$ as in Example 2.7, we recover the form (16). The monotonicity of the logarithm implies that ranks of the x_i and y_i are unchanged. Equation (17) in the case $k = 1$ is the subject of extensive conjectures of Ladas [9, 7]. In the following, we extend Ladas's max-type conjectures to the context of general rank-type equations. We state them in additive form (16), though they are easily translated to the multiplicative form (17).

Definition The solution $\{x_n\}$ of a difference equation is called *eventually periodic with period* p if there exists an integer $N > 0$ such that $x_{n+p} = x_n$ for all $n \geq N$.

Conjecture 4.2 *Consider the difference equation (16) where the* $b_1, \ldots, b_p$ *are ordered as*

$$b_{i_1} > \ldots > b_{i_{k-1}} > b_{i_k} > b_{i_{k+1}} > \ldots > b_{i_p},$$

that is, b_{i_k} *is the* k*th largest of distinct* b_i*. Then all solutions are eventually periodic with period* $2i_k$*. (The prime period may be a divisor of* $2i_k$*.)*

For example, the case $p = 3, k = 2$ concerns the equation

$$x_n = \text{median}\{-x_{n-1} + b_1, -x_{n-2} + b_2, -x_{n-3} + b_3\}. \tag{18}$$

The conjecture holds that for any initial conditions, the solution is eventually periodic with period

$$\begin{array}{lll} 2 & \text{if} & b_2 < b_1 < b_3 \text{ or } b_3 < b_1 < b_2, \\ 4 & \text{if} & b_1 < b_2 < b_3 \text{ or } b_3 < b_2 < b_1, \\ 6 & \text{if} & b_1 < b_3 < b_2 \text{ or } b_2 < b_3 < b_1. \end{array}$$

The requirement in Conjecture 4.2 that b_{i_k} is nonrepeating is important. If this requirement is lifted, although eventual periodicity is still expected, the formula for the period is more complicated. Continuing the case $p = 3, k = 2$ we have:

Proposition 4.3 *Uncountably many solutions of the equation (18) exist with prime periods*

$$\begin{array}{lll} 2 & \textit{if} & b_1 = b_3 \neq b_2, \\ 3 & \textit{if} & b_1 = b_2 \neq b_3, \\ 4 & \textit{if} & b_1 = b_2 = b_3, \\ 5 & \textit{if} & b_2 = b_3 \neq b_1. \end{array}$$

Proof It is easily checked that the following sequences satisfy the difference equation (18).

Case 1: $b_1 = b_3 \neq b_2$. For any β, define

$$\begin{aligned} x_1 &= \frac{b_1}{2} + \beta \\ x_2 &= \frac{b_1}{2} - \beta \end{aligned}$$

Then $x_1, x_2, x_1, x_2, \ldots$ is a solution.

Case 2: $b_1 = b_2 \neq b_3$. Set

$$\begin{aligned} x_1 &= \frac{b_1}{2} + \beta \\ x_2 &= \frac{b_1}{2} + \beta \\ x_3 &= \frac{b_1}{2} - \beta \end{aligned}$$

where β is any number between 0 and $(b_3 - b_1)/2$. Then $x_1, x_2, x_3, x_1, x_2, x_3, \ldots$ is a solution.

Case 3: $b_1 = b_2 = b_3$. For any β, set

$$\begin{aligned} x_1 &= \frac{b_1}{2} + \beta \\ x_2 &= \frac{b_1}{2} + \beta \\ x_3 &= \frac{b_1}{2} - \beta \\ x_4 &= \frac{b_1}{2} - \beta. \end{aligned}$$

Then $x_1, x_2, x_3, x_4, x_1, x_2, x_3, x_4, \ldots$ is a solution.

Case 4: $b_2 = b_3 \neq b_1$. Define

$$\begin{aligned}
x_1 &= \frac{b_1}{2} + \beta \\
x_2 &= \frac{b_1}{2} + \beta \\
x_3 &= \frac{b_1}{2} + \beta \\
x_4 &= \frac{b_1}{2} - \beta \\
x_5 &= \frac{b_1}{2} - \beta
\end{aligned}$$

where β is any number between 0 and $(b_1 - b_2)/2$. Then $x_1, x_2, x_3, x_4, x_5, x_1, x_2, x_3, x_4, x_5, \ldots$ is a solution, completing the proof.

We conjecture that the periodic solutions found above represent all possible periods for the $p = 3, k = 2$ rank-type equation. More precisely, we propose the following:

Conjecture 4.4 *Consider the difference equation (18). Then all solutions are eventually periodic, with period*

$$\begin{array}{lll}
2 & \textit{if} & b_2 < b_1 < b_3 \textit{ or } b_3 < b_1 < b_2, \\
4 & \textit{if} & b_1 < b_2 < b_3 \textit{ or } b_3 < b_2 < b_1, \\
6 & \textit{if} & b_1 < b_3 < b_2 \textit{ or } b_2 < b_3 < b_1, \\
2 & \textit{if} & b_1 = b_3 \neq b_2, \\
3 & \textit{if} & b_1 = b_2 \neq b_3, \\
4 & \textit{if} & b_1 = b_2 = b_3, \\
5 & \textit{if} & b_2 = b_3 \neq b_1.
\end{array}$$

See [16] for a proof of analogous results for the $p = 3, k = 1$ case. We expect that similar methods may suffice to prove Conjecture 4.4.

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