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Detection of periodic driving in nonautonomous difference equations

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Abstract.

An algorithm is proposed for determining the periodic behavior of the common driver of a system of nonautonomous difference equations from observations of the equation trajectories only. Methods of attractor reconstruction are used to build a semiconjugacy to a topological version of the driver system. The algorithm is described in detail and implemented on several examples.

§1. Introduction

System identification for nonlinear differential and difference equations has lagged far behind the linear case, due to the greater complexity of the task. Takens' Theorem [10, 1] of 25 years ago was a significant initial foray. The ramifications of this result, in an area that has become known as attractor reconstruction, continue to be worked out.

A particular case we will consider in this article is a system of driven difference equations of general form

$$\begin{aligned} x_t^1 &= f^1(x_{t-1}^1, \ldots, x_{t-n}^1, d_{t-1}) \\ &\vdots \\ x_t^k &= f^k(x_{t-1}^k, \ldots, x_{t-n}^k, d_{t-1}) \\ d_t &= g(d_{t-1}, \ldots, d_{t-p}) \end{aligned} \tag{1}$$

We often refer informally to the g system as the "driver" dynamics. Assume further that the equations $f^1, \ldots, f^k, g$ are unknown to us, and that we can only observe the outputs $x_i^1, \ldots, x_i^k$ as functions of time. We consider the problem of determining the dynamics of g alone.

Takens' Theorem discusses conditions under which such observations allow a topologically accurate reconstruction of the complete system

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dynamics. Such a reconstruction may not distinguish the g dynamics from the rest of the system. Our present goal goes beyond what is promised by Takens' Theorem, since our goal is to untangle the dynamics of g from the entire system. Moreover, if the theoretical obstructions can be overcome, it is reasonable to expect that the goal can be achieved with a far smaller data requirement than needed to reconstruct the entire system dynamics.

We will use make use of two recent extensions of Takens' Theorem. Stark [9] proved a reconstruction theorem for so-called skew product systems, in which an autonomous subsystem drives the rest of the system. This result was used in [8] to develop a driver reconstruction theorem. An algorithm based on this theorem is discussed below, along with two examples. In the first example, a set of five one-dimensional chaotic logistic maps are driven by a logistic map in a period-six window. The output of the five maps is used (without knowledge of the maps generating the output) to infer the periodicity of the driver. The second example demonstrates a similar reconstruction, using period-four forcing in a system of three two-dimensional difference equations.

It is hoped that this inquiry can further the development of computational techniques for system identification applied to network models of difference equations. The original motivation for this research was the identification of the dynamics of deep brain structures from simultaneous time series collected by surface electrodes. However, the question presents itself rather generally, whenever simultaneous driving of multiple, observable systems occurs.

§2. Background and theoretical results

To begin, we recall a reconstruction theorem for skew products due to Stark [9] (see also Casdagli [2]).

Theorem 1. *(Stark, 1999) Let D and X be compact manifolds, $dim(D) = d$, $dim(X) = k \geq 1$. Let $m \geq 2d + 2k + 1$, and assume the periodic orbits of period $< 2m$ of $g : D \to D$ are isolated and have distinct eigenvalues. Then there exists an open, dense set of C^1 functions $f : D \times X \to X$ and $h : X \to R$ for which the m-dimensional delay map is an embedding.*

The theorem states conditions under which the state space of a system consisting of the combined driver and response can be reconstructed. In the present article, the goal is to use the output of several response units to separate out the dynamics of the driver from the rest of the dynamics, as a way of breaking the system down into its component parts.

We will refer to the systems in (1) as $X_1, \ldots, X_k$ and D, respectively. Apply Stark's theorem to each $D \times X_i$ individually, where D is the ergodic attractor of the driving system and X_i is the state space of the ith nonautonomous system. According to the theorem, for generic dynamics and sufficiently large m, there is a one-to-one correspondence between m-tuples $(x_t^i, \ldots, x_{t-m+1}^i)$ and states of the dynamical attractor A_i in $D \times X_i$. In particular, as a consequence of the one-to-one correspondence guaranteed by Theorem 1, the equality

$$(2) \qquad (x_t^i, \ldots, x_{t-m+1}^i) = (x_{t'}^i, \ldots, x_{t'-m+1}^i)$$

implies that $d_t = d_{t'}$. This is the key to identifying individual states of the driver.

Due to continuity, (2) need not hold exactly to provide information. If the difference is small, say in the Euclidean norm, then d_t and $d_{t'}$ must also be close. This motivates choosing m sufficiently large so that Stark's theorem applies to $D \times X_i$ for $i = 1, \ldots, k$. We choose some $\epsilon > 0$ and for a given $(x_t^i, \ldots, x_{t-m+1}^i)$, group the set of times t' for which (2) holds within ϵ. In this way we form a set of equivalence classes.

To simplify the collection of the equivalence classes, one may choose one of the X_i and work in its delay coordinate space. This space contains a one-to-one representation of the attractor A_i in $D \times X_i$, according to Stark's theorem. Grouping the equivalence classes gives a quotient space of A_i, called D^*. Below, we see that D^* is a semiconjugacy with the dynamics of D.

There are three functions that can be defined for the set D^*. First, every equivalence class d^* in D^* has associated with it a unique d in D, so define the function s from D^* to D by $s(d^*) = d$. The function s is *onto*, meaning that the image of s is all of D. (This follows from the fact that the dynamics f is ergodic on D.) Second, there is a natural dynamical rule g^* from the set D^* to itself that is inherited from the dynamics on the delay coordinates. The equivalence class $g^*(d^*)$ is defined to be the one the elements of d^* are mapped to under the system dynamics f_1. In addition to the functions s and g^*, for each $1 \leq i \leq k$, the function p_i from A_i to D^* can be defined by sending each $a_i \in A_i$ to the equivalence class of a_i. The following diagram shows the relation between the functions p_i, s and the new dynamical system g^* on D^*.

$$\begin{tikzcd} D \arrow[d, "g"] & D^* \arrow[l, "s"'] \arrow[d, "g^*"] & A_i \arrow[l, "p_i"'] \arrow[d, "f_i^*"] \\ D & D^* \arrow[l, "s"'] & A_i \arrow[l, "p_i"'] \end{tikzcd}$$

Because g^* is onto, the right half of the diagram shows that g^* satisfies the definition of semiconjugacy. The map $g^* : D^* \to D^*$ is said to be *semiconjugate* to the map $f : D \to D$ if there exists an onto map $s : D^* \to D$ satisfying $g \circ s = s \circ g^*$, that is, the left side of the diagram commutes. The analogous statement about the right side of the diagram is also true. The following theorem was introduced in [8]:

Theorem 2. *(Shared Dynamics Reconstruction Theorem.) Assume $g : D \to D$ is ergodic, and in addition drives X_i, $1 \leq i \leq k$ as in (1). Choose m large enough and g, f^i generic such that all skew products $D \times X_i$ are reconstructed in R^m. Define $g^* : D^* \to D^*$ as the map induced as above. Then, generically, (1) the map g^* is semiconjugate to g, and (2) for each i, the induced map is semiconjugate to g^*.*

Roughly speaking, if g_1 is semiconjugate to g_2, then g_1 "contains" the dynamics of g_2. The content of the theorem is that according to the left hand square of the above diagram, D^* captures at least the dynamics of the driver D, and may contain more. However, according to the right side of the diagram for each i, any extra dynamics in D^* must be common to all of the X_i, due to part (2) of the theorem. This is the meaning of "shared dynamics".

Next we show how this theorem leads to an algorithm that extracts shared dynamics of the D using time series data observed from the X_i.

Shared Dynamics Algorithm.

Choose m large enough to unfold the dynamics on each $D \times X_i$, and use delay coordinates to create the reconstructed attractor A_i, which is in one-to-one correspondence with $D \times X_i$. Choose one of the X_i arbitrarily, say X_1. The basis of the algorithm is to group together points in X_1 that lie over the same point in D, the so-called fibers over D. According to the theorem, at each time t when the dynamics returns to the same point in X_j, the state of d in D is the same. With this information, one can search for delay vectors in X_j that are close in the delay reconstruction, and return information to X_1 about points over the same driver state d. A neighborhood size ε must be chosen to decide the meaning of nearly identical, for this purpose. The degree of discretization of the resulting dynamical attractor D^* will depend on this choice.

Using this method of associating points to the fibers over D, one proceeds through all points of the reconstructed attractor X_1 to fit them in an appropriate equivalence class. Choosing representatives for the equivalence classes from a chain of overlapping ε-neighborhoods retains the topological form imposed by the original dynamics. Note that no re-embedding is necessary, since the points of D^* constitute a subset of the reconstructed X_1, which has no self-intersections by assumption.

§3. Examples

In this section we illustrate the use of the Shared Dynamics Algorithm on two examples. In each case, a system of difference equations of form (1) is constructed and the output data x_i are provided to the algorithm for analysis.

Example 1. Consider the system

$$\begin{aligned} x_t^1 &= \lambda_1 x_{t-1}^1(1 - x_{t-1}^1) + 0.5 d_{t-1} \mod 1 \\ &\vdots \\ x_t^5 &= \lambda_5 x_{t-1}^5(1 - x_{t-1}^5) + 0.5 d_{t-1} \mod 1 \\ d_t &= \lambda_0 d_{t-1}(1 - d_{t-1}) \end{aligned} \tag{3}$$

where $\lambda_0 = 3.6266$ represents the period 6 window in the logistic bifurcation sequence, and $\lambda_1 = 3.77, \lambda_2 = 3.775, \lambda_3 = 3.78, \lambda_4 = 3.785, \lambda_5 = 3.79$ are chosen from a chaotic regime. The modulo function was used to keep trajectories from moving outside of the basin of the chaotic attractor.

Figure 1(a) shows the delay coordinate embedding of the reconstructed attractor A_1 of $D \times X_1$, using the embedding dimension $m = 3$. A trajectory of length 3000 time units is shown. The result of the Shared Dynamics Algorithm with $\epsilon = 0.005$, the reconstructed driver D^*, is shown in Figure 1(b). Each point represents an equivalence class of delay coordinate vectors. The topology and dynamics of the period-six attractor from D are efficiently recovered.

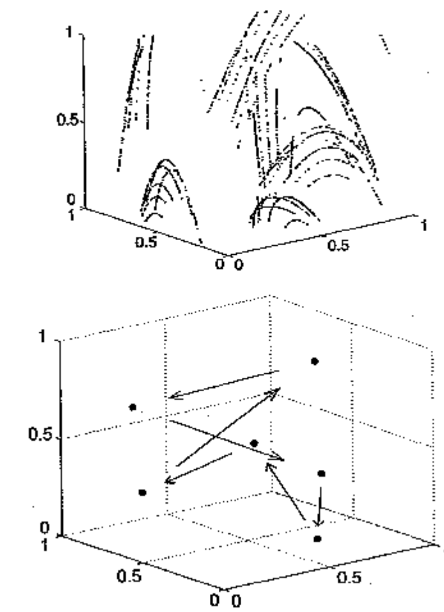

Fig. 1. (a) Delay coordinate reconstruction of one skew product from (3). (b) Reconstructed driver dynamics from the algorithm.

Example 2. The second example is the system

$$
\begin{aligned}
x_t^1 &= a_1 - \tfrac{1}{4}[(x_{t-1}^1)^2 + (y_{t-1}^1)^2] - \tfrac{1}{2}x_{t-1}^1 y_{t-1}^1 \\
&+ \frac{1+b}{2}x_{t-1}^1 + \frac{1-b}{2}y_{t-1}^1 + \alpha d_{t-1} \\
y_t^1 &= a_1 - \tfrac{1}{4}[(x_{t-1}^1)^2 + (y_{t-1}^1)^2] - \tfrac{1}{2}x_{t-1}^1 y_{t-1}^1 \\
&+ \frac{-1+b}{2}x_{t-1}^1 + \frac{-1-b}{2}y_{t-1}^1 \\
x_t^2 &= a_2 - \tfrac{1}{4}[(x_{t-1}^2)^2 + (y_{t-1}^2)^2] - \tfrac{1}{2}x_{t-1}^2 y_{t-1}^2 \\
&+ \frac{1+b}{2}x_{t-1}^2 + \frac{1-b}{2}y_{t-1}^2 + \alpha d_{t-1} \\
y_t^2 &= a_2 - \tfrac{1}{4}[(x_{t-1}^2)^2 + (y_{t-1}^2)^2] - \tfrac{1}{2}x_{t-1}^2 y_{t-1}^2 \\
&+ \frac{-1+b}{2}x_{t-1}^2 + \frac{-1-b}{2}y_{t-1}^2 \\
x_t^3 &= a_3 - \tfrac{1}{4}[(x_{t-1}^3)^2 + (y_{t-1}^3)^2] - \tfrac{1}{2}x_{t-1}^3 y_{t-1}^3 \\
&+ \frac{1+b}{2}x_{t-1}^3 + \frac{1-b}{2}y_{t-1}^3 + \alpha d_{t-1} \\
y_t^3 &= a_3 - \tfrac{1}{4}[(x_{t-1}^3)^2 + (y_{t-1}^3)^2] - \tfrac{1}{2}x_{t-1}^3 y_{t-1}^3 \\
&+ \frac{-1+b}{2}x_{t-1}^3 + \frac{-1-b}{2}y_{t-1}^3 \\
d_t &= a_0 - d_{t-1}^2 - b_0 d_{t-2}
\end{aligned}
\tag{4}
$$

where $a_0 = 0.95, b_0 = 0.4$ generates a period 4 attractor for the Hénon map [3], and $a_1 = 1.26, a_2 = 1.27, a_3 = 1.28, b = 0.3$ represent chaotic trajectories. The drive parameter is $\alpha = 0.05$.

Figure 2(a) shows the delay coordinate embedding of the reconstructed attractor A_1 of $D \times X_1$, using the embedding dimension $m = 3$ and delay coordinate vectors of form $(x_t^1, x_{t-1}^1, x_{t-2}^1)$. These vectors together with vectors of form $(x_t^i, x_{t-1}^i, x_{t-2}^i)$ for $i = 2, 3$ were used in the algorithm. A trajectory of length 2000 time units is shown. The result of the Shared Dynamics Algorithm with $\epsilon = 0.04$, the reconstructed driver D^*, is shown in Figure 2(b). Each point represents an equivalence class of delay coordinate vectors. The topology and dynamics of the period-four attractor from D are efficiently recovered.

§4. Conclusions

This article demonstrates a type of signal processing for difference equations, the idea being to identify system characteristics from output

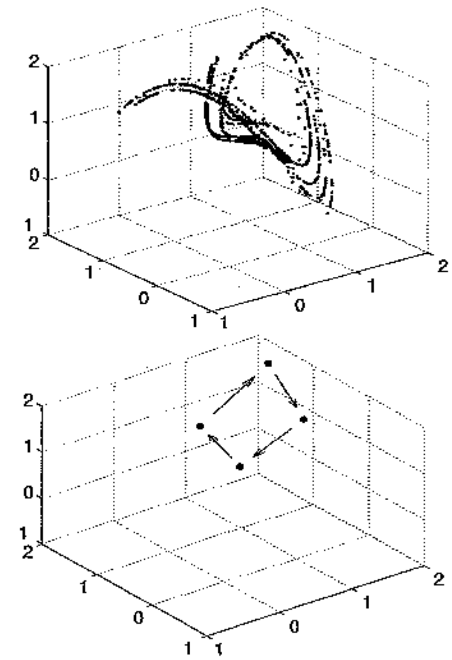

Fig. 2. (a) Reconstruction of one skew product from (4). (b) Reconstructed driver dynamics.

data. The specific goal in this case is to identify the dynamics of the common driver in a system of nonautonomous difference equations.

The examples provided deal with periodic drivers, such as the periodic six orbit driving five logistic maps in Example 1. The algorithm is not limited to periodic drivers, although the job of identifying the result will be more difficult the more complicated the driver dynamics. If the driver is chaotic, for example, sophisticated system identification methods may be needed to analyze and classify the result.

In the case of a chaotic system driving other chaotic systems, for example, although the algorithm introduced here will work in principle, the data requirements may be challenging. For periodic driving, we have shown in the examples that a few thousand data points suffice to determine the driver.

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