Numerical Methods
with Matlab
Tim Sauer

Contents

CHAPTER 0. Fundamentals ........................................... 1
  0.1 Evaluating a polynomial ...................................... 1
  0.2 Binary numbers ............................................. 5
    0.2.1 Decimal to binary...................................... 5
    0.2.2 Binary to decimal ..................................... 6
  0.3 Floating point representation of real numbers .............. 7
    0.3.1 Double precision floating point representation ....... 7
    0.3.2 Machine representation ................................ 11
    0.3.3 Addition of floating point numbers ................... 12
  0.4 Loss of significance ....................................... 14
  0.5 Review of calculus ....................................... 18

A Appendix: Matrix Algebra ..................................... 21
  A.1 Matrix fundamentals ....................................... 21
  A.2 Block multiplication ....................................... 22
  A.3 Eigenvalues and eigenvectors ............................... 23
  A.4 Symmetric matrices ....................................... 24

B Appendix: Introduction to Matlab ............................ 26
  B.1 Starting Matlab ........................................... 26
  B.2 Matlab graphics ........................................... 27
  B.3 Programming in Matlab .................................... 29
  B.4 Flow control .............................................. 29
  B.5 Functions ................................................ 30
  B.6 Matrix operations ......................................... 31

CHAPTER 1. Solving Equations .................................... 33
  1.1 What is an equation? ....................................... 33
    1.1.1 What is a solution? .................................... 33
    1.1.2 What is an equation, revisited ....................... 34
    1.1.3 The Bisection Method ................................ 35
    1.1.4 How accurate and how fast? .......................... 38
    1.1.5 The balloon problem ................................ 39
  1.2 Fixed point iteration ...................................... 41
    1.2.1 Fixed points of a function ............................ 42
    1.2.2 Geometry of Fixed Point Iteration ................. 44
    1.2.3 Linear Convergence of Fixed Point Iteration ....... 45
    1.2.4 Stopping criteria .................................... 51
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2.5</td>
<td>Basins of attraction for Fixed Point Iteration</td>
<td>52</td>
</tr>
<tr>
<td>1.3</td>
<td>Errors and Magnification</td>
<td>55</td>
</tr>
<tr>
<td>1.3.1</td>
<td>The limits of accuracy</td>
<td>55</td>
</tr>
<tr>
<td>1.3.2</td>
<td>Error magnification</td>
<td>60</td>
</tr>
<tr>
<td>1.4</td>
<td>Newton’s Method</td>
<td>63</td>
</tr>
<tr>
<td>1.4.1</td>
<td>Quadratic convergence of Newton’s method</td>
<td>65</td>
</tr>
<tr>
<td>1.4.2</td>
<td>Linear convergence of Newton’s method</td>
<td>67</td>
</tr>
<tr>
<td>1.5</td>
<td>Root-finding without derivatives</td>
<td>73</td>
</tr>
<tr>
<td>1.5.1</td>
<td>Secant method and variants</td>
<td>74</td>
</tr>
<tr>
<td>1.5.2</td>
<td>Brent’s Method</td>
<td>77</td>
</tr>
<tr>
<td>2.1</td>
<td>Gauss elimination</td>
<td>82</td>
</tr>
<tr>
<td>2.1.1</td>
<td>Operation counts</td>
<td>83</td>
</tr>
<tr>
<td>2.2</td>
<td>The LU factorization</td>
<td>88</td>
</tr>
<tr>
<td>2.2.1</td>
<td>Backsolving with the LU factorization</td>
<td>91</td>
</tr>
<tr>
<td>2.2.2</td>
<td>Complexity of the LU factorization</td>
<td>93</td>
</tr>
<tr>
<td>2.3</td>
<td>Sources of error</td>
<td>95</td>
</tr>
<tr>
<td>2.3.1</td>
<td>Error magnification and condition number</td>
<td>96</td>
</tr>
<tr>
<td>2.3.2</td>
<td>Swamping</td>
<td>99</td>
</tr>
<tr>
<td>2.4</td>
<td>Partial pivoting and the PA=LU factorization</td>
<td>102</td>
</tr>
<tr>
<td>2.4.1</td>
<td>Permutation matrices</td>
<td>104</td>
</tr>
<tr>
<td>2.4.2</td>
<td>$PA = LU$ factorization</td>
<td>105</td>
</tr>
<tr>
<td>2.4.3</td>
<td>Matlab commands for linear systems</td>
<td>108</td>
</tr>
<tr>
<td>2.5</td>
<td>Iterative methods</td>
<td>110</td>
</tr>
<tr>
<td>2.5.1</td>
<td>Jacobi method</td>
<td>110</td>
</tr>
<tr>
<td>2.5.2</td>
<td>Gauss-Seidel Method and SOR</td>
<td>113</td>
</tr>
<tr>
<td>2.5.3</td>
<td>Convergence of iterative methods</td>
<td>114</td>
</tr>
<tr>
<td>2.5.4</td>
<td>Sparse matrix computations</td>
<td>116</td>
</tr>
<tr>
<td>2.6</td>
<td>Conjugate gradient method</td>
<td>120</td>
</tr>
<tr>
<td>2.6.1</td>
<td>Positive definite matrices</td>
<td>120</td>
</tr>
<tr>
<td>2.6.2</td>
<td>The conjugate gradient iteration</td>
<td>121</td>
</tr>
<tr>
<td>2.7</td>
<td>Nonlinear systems of equations</td>
<td>125</td>
</tr>
<tr>
<td>2.7.1</td>
<td>Multivariable Newton’s method</td>
<td>125</td>
</tr>
<tr>
<td>3.1</td>
<td>Data and interpolating functions</td>
<td>131</td>
</tr>
<tr>
<td>3.1.1</td>
<td>Lagrange interpolation</td>
<td>132</td>
</tr>
<tr>
<td>3.1.2</td>
<td>Newton’s divided differences</td>
<td>134</td>
</tr>
<tr>
<td>3.1.3</td>
<td>How many degree $d$ polynomials pass through $n$ points?</td>
<td>137</td>
</tr>
<tr>
<td>3.1.4</td>
<td>Code for interpolation</td>
<td>138</td>
</tr>
<tr>
<td>3.1.5</td>
<td>Representing functions by approximating polynomials</td>
<td>140</td>
</tr>
<tr>
<td>3.2</td>
<td>Interpolation error</td>
<td>144</td>
</tr>
<tr>
<td>3.2.1</td>
<td>Interpolation error formula</td>
<td>144</td>
</tr>
<tr>
<td>3.2.2</td>
<td>Proof of Newton form and error formula</td>
<td>146</td>
</tr>
<tr>
<td>3.2.3</td>
<td>Runge phenomenon</td>
<td>148</td>
</tr>
<tr>
<td>Chapter</td>
<td>Section</td>
<td>Title</td>
</tr>
<tr>
<td>---------</td>
<td>---------</td>
<td>-------------------------------</td>
</tr>
<tr>
<td>3.3</td>
<td>3.3.1</td>
<td>Chebyshev’s Theorem</td>
</tr>
<tr>
<td>3.3</td>
<td>3.3.2</td>
<td>Chebyshev polynomials.</td>
</tr>
<tr>
<td>3.3</td>
<td>3.3.3</td>
<td>Change of interval</td>
</tr>
<tr>
<td>3.4</td>
<td>3.4.1</td>
<td>Cubic splines</td>
</tr>
<tr>
<td>3.4</td>
<td>3.4.2</td>
<td>Calculation of the cubic spline.</td>
</tr>
<tr>
<td>3.4</td>
<td>3.4.3</td>
<td>Endpoint conditions</td>
</tr>
<tr>
<td>3.4</td>
<td>3.4.4</td>
<td>Bezier splines</td>
</tr>
<tr>
<td>4</td>
<td>4.1</td>
<td>Inconsistent systems of equations.</td>
</tr>
<tr>
<td>4</td>
<td>4.2</td>
<td>Modeling data by least squares.</td>
</tr>
<tr>
<td>4</td>
<td>4.3</td>
<td>A survey of models.</td>
</tr>
<tr>
<td>4.3</td>
<td>4.3.1</td>
<td>Periodic data</td>
</tr>
<tr>
<td>4.3</td>
<td>4.3.2</td>
<td>Linearizing the model.</td>
</tr>
<tr>
<td>5</td>
<td>5.1</td>
<td>Numerical differentiation</td>
</tr>
<tr>
<td>5</td>
<td>5.1.1</td>
<td>Difference formulas</td>
</tr>
<tr>
<td>5</td>
<td>5.1.2</td>
<td>Rounding error</td>
</tr>
<tr>
<td>5</td>
<td>5.1.3</td>
<td>Extrapolation</td>
</tr>
<tr>
<td>5</td>
<td>5.1.4</td>
<td>Symbolic differentiation and integration.</td>
</tr>
<tr>
<td>5</td>
<td>5.2</td>
<td>Newton-Cotes formulas for numerical integration</td>
</tr>
<tr>
<td>5</td>
<td>5.2.1</td>
<td>Three simple integrals for Newton-Cotes Formulas.</td>
</tr>
<tr>
<td>5</td>
<td>5.2.2</td>
<td>Trapezoid rule</td>
</tr>
<tr>
<td>5</td>
<td>5.2.3</td>
<td>Simpson’s Rule</td>
</tr>
<tr>
<td>5</td>
<td>5.2.4</td>
<td>Composite Newton-Cotes Formulas</td>
</tr>
<tr>
<td>5</td>
<td>5.2.5</td>
<td>Open Newton-Cotes methods</td>
</tr>
<tr>
<td>5</td>
<td>5.3</td>
<td>Romberg integration</td>
</tr>
<tr>
<td>5</td>
<td>5.4</td>
<td>Adaptive quadrature</td>
</tr>
<tr>
<td>5</td>
<td>5.5</td>
<td>Gaussian quadrature</td>
</tr>
<tr>
<td>6</td>
<td>6.1</td>
<td>Initial value problems</td>
</tr>
<tr>
<td>6</td>
<td>6.1.1</td>
<td>Euler’s method.</td>
</tr>
<tr>
<td>6</td>
<td>6.1.2</td>
<td>Existence, uniqueness, and continuity for solutions.</td>
</tr>
<tr>
<td>6</td>
<td>6.1.3</td>
<td>First-order linear equations</td>
</tr>
<tr>
<td>6.2</td>
<td>Analysis of IVP solvers.</td>
<td>240</td>
</tr>
<tr>
<td>6.2.1</td>
<td>Local and global truncation error</td>
<td>241</td>
</tr>
<tr>
<td>6.2.2</td>
<td>The trapezoid method</td>
<td>245</td>
</tr>
<tr>
<td>6.2.3</td>
<td>Taylor methods</td>
<td>248</td>
</tr>
<tr>
<td>6.3</td>
<td>Systems of ordinary differential equations.</td>
<td>250</td>
</tr>
<tr>
<td>6.3.1</td>
<td>Higher order equations</td>
<td>252</td>
</tr>
<tr>
<td>6.3.2</td>
<td>The pendulum</td>
<td>253</td>
</tr>
<tr>
<td>6.3.3</td>
<td>Orbital mechanics</td>
<td>256</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>6.4</td>
<td>Runge-Kutta methods and applications</td>
<td>261</td>
</tr>
<tr>
<td>6.4.1</td>
<td>Classical examples</td>
<td>263</td>
</tr>
<tr>
<td>6.5</td>
<td>Variable step-size methods</td>
<td>267</td>
</tr>
<tr>
<td>6.6</td>
<td>Implicit methods and stiff equations</td>
<td>274</td>
</tr>
<tr>
<td>6.7</td>
<td>Multistep methods</td>
<td>278</td>
</tr>
<tr>
<td>6.7.1</td>
<td>Generating multistep methods</td>
<td>280</td>
</tr>
<tr>
<td>6.7.2</td>
<td>Explicit multistep methods</td>
<td>281</td>
</tr>
<tr>
<td>6.7.3</td>
<td>Implicit multistep methods</td>
<td>285</td>
</tr>
<tr>
<td>7.1</td>
<td>Solutions of boundary value problems</td>
<td>291</td>
</tr>
<tr>
<td>7.1.1</td>
<td>Shooting method</td>
<td>294</td>
</tr>
<tr>
<td>7.2</td>
<td>Finite difference methods</td>
<td>296</td>
</tr>
<tr>
<td>7.2.1</td>
<td>Derivative formulas</td>
<td>296</td>
</tr>
<tr>
<td>7.2.2</td>
<td>Nonlinear boundary value problems</td>
<td>299</td>
</tr>
<tr>
<td>7.3</td>
<td>Collocation and the Finite Element Method</td>
<td>303</td>
</tr>
<tr>
<td>7.3.1</td>
<td>Collocation</td>
<td>303</td>
</tr>
<tr>
<td>7.3.2</td>
<td>Finite elements and the Galerkin method</td>
<td>305</td>
</tr>
<tr>
<td>8.1</td>
<td>Parabolic equations</td>
<td>311</td>
</tr>
<tr>
<td>8.1.1</td>
<td>Forward difference method</td>
<td>312</td>
</tr>
<tr>
<td>8.1.2</td>
<td>Stability analysis of forward difference method</td>
<td>315</td>
</tr>
<tr>
<td>8.1.3</td>
<td>Backward difference method</td>
<td>317</td>
</tr>
<tr>
<td>8.1.4</td>
<td>Crank-Nicolson method</td>
<td>320</td>
</tr>
<tr>
<td>8.2</td>
<td>Hyperbolic equations</td>
<td>325</td>
</tr>
<tr>
<td>8.3</td>
<td>Elliptic equations</td>
<td>328</td>
</tr>
<tr>
<td>8.3.1</td>
<td>Finite difference methods for elliptic equations</td>
<td>329</td>
</tr>
<tr>
<td>8.3.2</td>
<td>Finite element method for elliptic equations</td>
<td>333</td>
</tr>
<tr>
<td>9.1</td>
<td>Random numbers</td>
<td>341</td>
</tr>
<tr>
<td>9.1.1</td>
<td>Pseudo-random numbers</td>
<td>342</td>
</tr>
<tr>
<td>9.2</td>
<td>Monte-Carlo simulation</td>
<td>348</td>
</tr>
<tr>
<td>9.2.1</td>
<td>Power laws for Monte Carlo estimation</td>
<td>348</td>
</tr>
<tr>
<td>9.2.2</td>
<td>Quasi-random numbers</td>
<td>350</td>
</tr>
<tr>
<td>9.3</td>
<td>Discrete and continuous Brownian motion</td>
<td>355</td>
</tr>
<tr>
<td>9.3.1</td>
<td>Random walks</td>
<td>355</td>
</tr>
<tr>
<td>9.3.2</td>
<td>Continuous Brownian motion</td>
<td>357</td>
</tr>
<tr>
<td>9.4</td>
<td>Stochastic differential equations</td>
<td>359</td>
</tr>
<tr>
<td>9.4.1</td>
<td>Adding noise to ODEs</td>
<td>360</td>
</tr>
<tr>
<td>9.4.2</td>
<td>Numerical methods for SDEs</td>
<td>363</td>
</tr>
</tbody>
</table>
# CONTENTS

## CHAPTER 10. Trigonometric Interpolation and the FFT

10.1 The Fourier Transform ........................................ 371
  10.1.1 Complex arithmetic .......................... 371
  10.1.2 Discrete Fourier Transform ......................... 374
  10.1.3 The Fast Fourier Transform ......................... 377
10.2 Trigonometric interpolation .................................. 380
  10.2.1 The DFT Interpolation Theorem .................... 380
  10.2.2 Orthogonality and interpolation .................... 385
  10.2.3 Least squares fitting with trigonometric functions 387
  10.2.4 Sound, noise, and filtering ........................ 392

## CHAPTER 11. Compression

11.1 The Discrete Cosine Transform .............................. 397
  11.1.1 One-dimensional DCT ............................... 397
11.2 Two-dimensional DCT and image compression ................ 401
  11.2.1 The two-dimensional discrete cosine transform ... 401
  11.2.2 Image compression ................................ 405
  11.2.3 Quantization of image transforms .................. 408
11.3 Modified DCT and sound compression ........................ 413
  11.3.1 Modified Discrete Cosine Transform ............... 413
  11.3.2 Quantization ..................................... 418
  11.3.3 Fast Cosine Transforms ............................ 422

## CHAPTER 12. Eigenvalues and Singular Values

12.1 Power iteration methods .................................... 429
  12.1.1 Power iteration ................................... 430
  12.1.2 Convergence of power iteration ................... 431
  12.1.3 Inverse power iteration ........................... 432
12.2 QR algorithm ............................................. 434
  12.2.1 Simultaneous iteration ............................ 434
  12.2.2 Real Schur form and QR .......................... 437
  12.2.3 Householder reflectors .............................. 439
  12.2.4 Upper Hessenberg form ............................. 443
12.3 Singular value decomposition ................................ 449
  12.3.1 Finding the SVD in general ......................... 452
  12.3.2 Special case: symmetric matrices ................. 454
12.4 Applications of the SVD .................................... 455
  12.4.1 Properties of the SVD ............................. 455
  12.4.2 Dimension reduction ............................... 457
  12.4.3 Compression ....................................... 458
  12.4.4 Calculating the SVD ............................... 459

## CHAPTER 13. Optimization

13.1 Unconstrained optimization without derivatives ............ 463
  13.1.1 Golden section search .............................. 464
  13.1.2 Successive parabolic interpolation ................. 467
13.1.3 Nelder-Mead search. 469
13.2 Unconstrained optimization with derivatives. 472
  13.2.1 Newton’s method. 472
  13.2.2 Steepest descent. 474
  13.2.3 Conjugate gradient search. 475
  13.2.4 Nonlinear least squares. 476