

## On Fixed Point Stability for Set-Valued Contractive Mappings with Applications to Generalized Differential Equations

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Let  $(X, d)$  be a metric space and  $C(X)$  the family of nonempty closed subsets of  $X$ . For  $A, B$  in  $C(X)$ , we shall denote  $D(A, B)$  the (extended real-valued) Hausdorff distance between  $A$  and  $B$ , i.e.,

$$D(A, B) = \max \left\{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \right\},$$

where  $d(x, C) = \inf_{c \in C} d(x, c)$ . A set-valued mapping  $T: X \rightarrow C(X)$  is a  $\lambda$ -contraction if  $0 \leq \lambda < 1$  and

$$D(Tx, Ty) \leq \lambda d(x, y)$$

for  $x, y \in X$ . If  $X$  is complete, then every set-valued contraction has a fixed point, i.e., a point  $x$  with  $x \in Tx$ . The set of fixed points of  $T$  will be denoted by  $F(T)$ .

Stability of fixed points of set-valued contractions was investigated in [12] and [10]. In [10], a stability theorem was proved under some rather restrictive conditions, including (i) the domain of the maps being a closed convex-bounded subset of a Hilbert space, (ii) the image of each point under each map being a closed convex subset. Theorem 1 below removes these conditions.

**LEMMA 1.** *Let  $X$  be a complete metric space and  $T_1$  and  $T_2$  be  $\lambda$ -contractions from  $X$  into  $C(X)$ . Then*

$$D(F(T_1), F(T_2)) \leq \frac{1}{1-\lambda} \sup_{x \in X} D(T_1(x), T_2(x)).$$

*Proof.* Write  $K = \sup_{x \in X} D(T_1(x), T_2(x))$ . We may assume that  $K < \infty$ . Let  $\varepsilon > 0$  be arbitrary. Choose  $c > 0$  such that

$$c \sum_{n=1}^{\infty} n\lambda^n < 1$$

and set  $\varepsilon_1 = (c/(1-\lambda))\varepsilon$ .

Let  $x_0 \in F(T_1)$ . Since  $D(T_1(x_0), T_2(x_0)) \leq K$ , we may choose  $x_1 \in T_2(x_0)$  with  $d(x_1, x_0) \leq K + c$ . Also from  $D(T_2(x_1), T_2(x_0)) \leq \lambda d(x_1, x_0)$ , we may choose  $x_2 \in T_2(x_1)$  with  $d(x_2, x_1) \leq \lambda d(x_1, x_0) + \lambda\varepsilon_1$ . By induction, we can define a sequence  $x_n$  such that  $x_{n+1} \in T_2(x_n)$  and

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}) + \lambda^n \varepsilon_1$$

for  $n \geq 1$ . We have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \lambda d(x_n, x_{n-1}) + \lambda^n \varepsilon_1 \\ &\leq \lambda(\lambda d(x_{n-1}, x_{n-2}) + \lambda^{n-1} \varepsilon_1) + \lambda^n \varepsilon_1 \\ &= \lambda^2 d(x_{n-1}, x_{n-2}) + 2\lambda^n \varepsilon_1 \\ &\dots \end{aligned}$$

and

$$d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0) + n\lambda^n \varepsilon_1,$$

valid also for  $n=0$ . Thus

$$\sum_{n=m}^{\infty} d(x_{n+1}, x_n) \leq \frac{\lambda^m}{1-\lambda} d(x_1, x_0) + \sum_{n=m}^{\infty} n\lambda^n \varepsilon_1 \rightarrow 0$$

as  $m \rightarrow \infty$ . Thus  $x_n$  is a Cauchy sequence with a limit  $\bar{x}$ . By the continuity of  $T_2$ ,  $T_2(x_n) \rightarrow T_2(\bar{x})$  in the Hausdorff metric. Moreover, since  $x_{n+1} \in T_2 x_n$ , we have  $\bar{x} \in T_2(\bar{x})$ , i.e.,  $\bar{x} \in F(T_2)$ . Furthermore

$$\begin{aligned} d(x_0, \bar{x}) &\leq \sum_{n=0}^{\infty} d(x_{n+1}, x_n) \\ &\leq \frac{1}{1-\lambda} d(x_1, x_0) + \sum_{n=1}^{\infty} n\lambda^n \varepsilon_1 \\ &\leq \frac{1}{1-\lambda} (d(x_1, x_0) + \varepsilon) \\ &\leq \frac{1}{1-\lambda} (K + 2\varepsilon). \end{aligned} \tag{1}$$

By symmetry, we also have: for every  $x_0 \in F(T_2)$ , there exists  $\bar{x} \in F(T_1)$  such that (1) holds. Hence

$$D(F(T_1), F(T_2)) \leq \frac{1}{1-\lambda} K.$$

**THEOREM 1.** *Let  $X$  be a complete metric space and  $T_i: X \rightarrow C(X)$  a sequence of  $\lambda$ -contractions  $i=0, 1, 2, \dots$ . If  $\lim_{i \rightarrow \infty} D(T_i(x), T_0(x)) = 0$  uniformly for all  $x$  in  $X$ , then  $\lim_{i \rightarrow \infty} D(F(T_i), F(T_0)) = 0$ .*

*Proof.* For  $\varepsilon > 0$  choose  $N$  such that  $\sup_{x \in X} D(T_i(x), T_0(x)) < (1-\lambda)\varepsilon$  for  $i \geq N$ . Then  $D(F(T_i), F(T_0)) < \varepsilon$  for  $i \geq N$  by Lemma 1.

*Remark 1.* It is generally not true that  $F(T_i) \rightarrow F(T_0)$  if  $T_i$  do not have a uniform Lipschitz constant  $\lambda < 1$ . For a simple example, let  $T$  be a nonexpansive selfmapping of a closed convex-bounded subset  $C$  of a Banach space with more than one fixed point and let  $T_i x = (1/i)x_0 + (1-1/i)Tx$ , where  $x_0$  is a fixed element of  $C$  and  $T_0 = T$ .

A Banach space is said to satisfy *Opial's condition* if  $x_n \rightarrow x_0$  implies

$$\liminf \|x_n - x_0\| < \liminf \|x_n - x\|$$

for  $x \neq x_0$ . Here  $\rightarrow$  denotes the weak convergence. A map  $J$  of a Banach space  $X$  into its dual  $X^*$  is a duality map if  $(x, J(x)) = \|J(x)\| \|x\|$  and  $\|J(x)\| = \mu(\|x\|)$  for  $x \in X$ , where  $\mu$  is a nonnegative, nondecreasing function on  $R^1$  with  $\mu(0) = 0$ . The map  $J$  is said to be weakly continuous if it is continuous on  $X$  with the weak topology into  $X^*$  with the weak\* topology. It is known that a Banach space satisfies Opial's condition if it admits a weakly continuous duality map, but not conversely, see [4].

A set-valued map in a Banach space is said to be nonexpansive if  $D(Tx, Ty) \leq \|x - y\|$ . In [9], a stability theorem was proved for nonexpansive maps in a Banach space that is strictly convex and admits a weakly continuous duality map. We prove in the following the same theorem under the weaker assumption of Opial's condition, thus removing the strict convexity also. Note that we also weaken the conditions on the domains and the range of  $T_i$ , in particular,  $T_i(x)$  need not be convex and  $T_i$  need not be defined on the whole space  $X$ . In the following,  $K(X)$  will denote the family of compact nonempty subsets of  $X$ .

**THEOREM 2.** *Let  $X$  be a Banach space satisfying Opial's condition,  $B$  a weakly closed subset of  $X$ , and  $T$  a mapping of  $B$  into  $C(X)$ . Assume that  $(T_i)$  is a sequence of nonexpansive maps of  $B$  into  $K(X)$  converging pointwise to  $T$  in the Hausdorff metric  $D$ . If  $x_i$  is a fixed point of  $T_i$  in  $B$  for  $i=1, 2, \dots$  and  $x_i \rightarrow x_0$ , then  $x_0$  is a fixed point of  $T$ .*

*Proof.*  $T(x_0)$ , being the limit of the compact sets  $T_i(x_0)$ , is compact.

Let  $\varepsilon_i = D(T_i(x_0), T(x_0))$ . For each  $i$ , there exist, by compactness,  $y_i \in T_i(x_0)$  and  $z_i \in T(x_0)$  such that

$$\|x_i - y_i\| \leq D(T_i(x_i), T_i(x_0)) \leq \|x_i - x_0\|$$

and

$$\|y_i - z_i\| \leq \varepsilon_i.$$

So the sequence  $z_i$  has a convergent subsequence which we will still denote by  $z_i$ . If  $z_i \rightarrow z_0$ , then

$$\begin{aligned} \|x_i - z_0\| &\leq \|x_i - z_i\| + \|z_i - z_0\| \\ &\leq \|x_i - y_i\| + \|y_i - z_i\| + \|z_i - z_0\| \\ &\leq \|x_i - x_0\| + \varepsilon_i + \|z_i - z_0\|. \end{aligned}$$

Thus,

$$\liminf \|x_i - z_0\| \leq \liminf \|x_i - x_0\|.$$

By Opial's condition we must have  $x_0 = z_0 \in T(x_0)$ . ■

*Remark 2.* Existence of fixed points for multivalued nonexpansive mappings in Banach spaces satisfying the Opial's condition was previously proved by Lami-Dozo [7].

*Remark 3.* A bounded sequence  $x_n$  is said to be almost convergent (see [5]; also called  $\mathcal{A}$ -convergent in [8]) if every subsequence of  $x_n$  has the same asymptotic center. A space satisfies Opial's condition if and only if weak convergence and almost convergence are identical. (Note that  $\liminf$  could be replaced by  $\limsup$  in Opial's condition.) With about the same proof of Theorem 2, we have the following.

**THEOREM 3.** *Let  $X$  be a uniformly convex Banach space,  $B$  a closed convex bounded subset of  $X$  and  $T_0$  a mapping of  $B$  into  $C(B)$ . Assume that  $\{T_i\}$  is a sequence of nonexpansive maps of  $B$  into  $K(B)$  converging pointwise to  $T_0$  in the Hausdorff metric  $D$ . If  $x_i$  is a fixed point of  $T_i$  for  $i = 1, 2, \dots$  and  $x_i$  almost converges to  $x_0$ , then  $x_0$  is a fixed point of  $T_0$ .*

*Remark 4.* It was shown in [5, 8] that every sequence in  $B$  above has an almost convergent subsequence (w.r.t.  $B$ ).

We now give an application of Theorem 1 to problems on the stability of solution sets for generalized differential equations. The existence of a solution to the initial value problem

$$\dot{x}(t) \in R(t, x(t)), \quad x(0) = x_0 \quad (2)$$

where  $R(t, x)$  is a compact convex subset of Euclidean  $n$ -space  $\mathbb{R}^n$  for real  $t$  and  $x \in \mathbb{R}^n$ , was proved by Filippov ([3, Theorem 1] or [6, Theorem 2.1])

and Castaing [2] under certain conditions on  $R$ . In [11], Markin proved a stability theorem on the set of solutions to (2) using the  $L^2$  norm. We proved in the following a similar theorem using the sup norm.

Let  $B$  be an origin-centered closed ball in  $\mathbb{R}^n$  and  $CC(B)$  the family of nonempty closed convex subsets of  $B$  with Hausdorff metric  $D$  generated by the norm  $\|\cdot\|$  of  $\mathbb{R}^n$ . Let  $C[0, a]$  be the continuous maps of  $[0, a]$  into  $\mathbb{R}^n$  with the sup norm. A solution to (2) will be any absolutely continuous map  $y$  with  $\dot{y}(t) \in R(t, y(t))$  a.e. and  $y(0) = x_0$ . Assume that  $R$  is a continuous map of  $[0, a] \times B$  into  $CC(B)$  satisfying

$$D(R(t, x), R(t, y)) \leq k \|x - y\|$$

for some  $k > 0$ . For  $b \in B$ , we will denote  $S(b)$  the set of solutions of (2) on  $[0, a]$  with  $x_0 = b$ .  $S(b)$  is nonempty ([6], Theorem 2.1) and compact [2].

**THEOREM 4.** *If  $ka < 1$ , then  $S(b)$  is continuous from  $B$  into the family of nonempty compact subsets of  $C[0, a]$  equipped with the Hausdorff metric.*

*Proof.* Suppose  $b_n \rightarrow b_0$ . For  $x \in C[0, a]$ , define

$$F(b, x) = \left\{ y \in C[0, a] : y(t) = b + \int_0^t z(s) ds, z(s) \in R(s, x(s)) \right\}.$$

Let  $T_n(x) = F(b_n, x)$ ,  $n = 0, 1, 2, \dots$ . Since  $T_0(x) = b_n - b_0 + T_n(x)$ , it is obvious that  $T_n(x)$  converges uniformly to  $T_0(x)$ .  $T_n(x)$  is compact convex for each  $x$  and  $n$  [1].

Given any pair  $x_1, x_2 \in C[0, a]$  and  $y_1 \in F(b, x_1)$ , let

$$y_1(t) = b + \int_0^t r_1(s) ds, \quad r_1(s) \in R(s, x_1(s)).$$

Define  $r_2(s)$  to be the point in  $R(s, x_2(s))$  nearest to  $r_1(s)$ , i.e.,  $r_2(s) \in R(s, x_2(s))$  and

$$\|r_1(s) - r_2(s)\| = \min \{ \|r_1(s) - z\| : z \in R(s, x_2(s)) \}.$$

It follows from the measurability of  $r_1(s)$  and the continuity of  $R(s, x_2(s))$  and the nearest point projection that  $r_2(s)$  is measurable.

Setting  $y_2(t) = b + \int_0^t r_2(s) ds$ , we have

$$\begin{aligned} \|y_2 - y_1\| &\leq \int_0^a \|r_1(s) - r_2(s)\| ds \leq k \int_0^a \|x_1(s) - x_2(s)\| ds \\ &\leq ka \|x_1 - x_2\|. \end{aligned}$$

Thus  $T_n$  are  $\lambda$ -contractions with  $\lambda = ka < 1$ . By Theorem 1,  $F(T_n) \rightarrow F(T_0)$ , i.e.,  $S(b_n) \rightarrow S(b_0)$ . ■

*Remark 5.* Note that what we need out of  $F(b, x)$  in the above proof in order that Theorem 1 can be applied is that it be closed (and nonempty).

*Remark 6.* Professor Stephen Saperstone has shown me that the proof of Theorem 3 can be modified to yield the following more general

**THEOREM.** For each  $n=0, 1, 2, \dots$ , let  $R_n$  be a continuous map of  $[0, a] \times B$  into  $C(B)$  satisfying

$$D(R_n(t, x), R_n(t, y)) \leq k \|x - y\|$$

for some  $k > 0$ . Assume that  $R_n \rightarrow R_0$  uniformly on  $[0, a] \times B$ . For each  $b \in B$  and  $n=0, 1, 2, \dots$ , let  $S_n(b)$  be the set of solutions of

$$\dot{x}(t) \in R_n(t, x(t)), \quad x(0) = b.$$

If  $ka < 1$  and  $b_n \rightarrow b_0$  in  $B$ , then  $S_n(b_n) \rightarrow S_0(b_0)$ .

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