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ON THE NORMAL STRUCTURE COEFFICIENT AND THE BOUNDED SEQUENCE COEFFICIENT

TECK-CHEONG LIM

Abstract. The two notions of normal structure coefficient and bounded sequence coefficient introduced by Bynum are shown to be the same. A lower bound for the normal structure coefficient in $L^p$, $p > 2$, is also given.

Let $X$ be a Banach space and $C$ a closed convex bounded subset of $X$. For each $x$ in $C$, let $R(x, C) = \sup \{ \| x - y \| : y \in C \}$ and let $R(C)$ denote the Chebyshev radius of $C$ [2, p. 178]:

$$R(C) = \inf \{ R(x, C) : x \in C \}.$$

Let $D(C)$ denote the diameter of $C$, $D(C) = \sup \{ \| x - y \| : x, y \in C \}$.

For a bounded sequence $\{ x_n \}$ in $X$, the asymptotic diameter $A(\{ x_n \})$ of $\{ x_n \}$ is defined to be $\lim_{n \to \infty} \sup \{ \| x_k - x_m \| : m \geq n, k \geq n \}$.

In [1], Bynum introduced the following two coefficients of $X$, called the normal structure coefficient and the bounded sequence coefficient respectively:

$$N(X) = \inf \{ \frac{D(C)}{R(C)} : C \text{ closed convex bounded nonempty subsets of } X \text{ with } |C| > 1 \},$$

$$BS(X) = \sup \left\{ M : \text{for every bounded sequence } \{ x_n \} \text{ in } X, \text{ there exists } y \in \overline{Co}(x_n) \text{ such that } M \lim_{n \to \infty} \sup \| x_n - y \| = A(\{ x_n \}) \right\}.$$
In [1], Bynum mentioned that the two coefficients \( N(X) \) and \( BS(X) \) are equal in a separable Banach space \( X \). In this note, we shall show that the three coefficients are equal in any Banach space \( X \).

**Theorem 1.** For a Banach space \( X \), \( N(X) = BS(X) = A(X) \).

**Proof.** It follows readily from the definition that \( BS(X) = A(X) \). Indeed, we may assume that the sequences in the definition of \( BS(X) \) are nonconvergent. Clearly \( BS(X) \leq A(X) \). On the other hand for each \( \lambda > 1 \), \( A(X)/\lambda \) belongs to the defining set of \( BS(X) \) and thus \( BS(X) \geq A(X) \). Bynum [1] proved that \( N(X) \leq BS(X) \). To prove that \( BS(X) \leq N(X) \), it suffices to show that for any bounded convex nonempty set \( C \) with more than one point, there is a separable closed convex subset \( C_1 \) such that \( R(C_1) = R(C) \). Indeed, if \( \{ x_n \} \) is a dense sequence in \( C_1 \) and \( M \) is a number in the defining set of \( BS(X) \), then

\[
M \leq A(\{ x_n \}) \leq \limsup \| x_n - y \| = \frac{D(C_1)}{R(y, C_1)} \leq \frac{D(C_1)}{R(C_1)} \leq \frac{D(C)}{R(C)}.
\]

To construct \( C_1 \), we start out with a sequence of points \( \{ z_n \} \) in \( C \) such that \( \lim_{n \to -\infty} R(z_n, C) = R(C) \). Let \( U_1 = \text{Co}(\{ z_n \}) \). Let \( V_1 = \{ x \in U_1 : R(x, U_1) < R(C) \} \) and let \( V_1 \) be a countable dense subset of \( V_1 \). For each \( x \) in \( V_1 \), let \( D_x \) be a sequence of points in \( C \) such that \( R(x, D_x) \geq R(C) \). Let \( X_1 \) be the countable subset \( \bigcup \{ D_x : x \in V_1 \} \) and \( U_2 = \text{Co}(U_1 \cup X_1) \). We define similarly \( V_2 \), \( W_2 \) and \( X_2 \) from \( U_2 \) and continue this process to obtain an increasing sequence of convex sets \( U_1 \subset U_2 \subset U_3 \subset \cdots \subset U_n \subset \cdots \). Let \( C_1 = \text{Co}(\bigcup U_n) \). \( C_1 \) is separable. Since \( R(z_n, C_1) \leq R(z_n, C) \) and \( \lim_{n \to -\infty} R(z_n, C) = R(C) \), we have \( R(C_1) \leq R(C) \). From the way \( U_n \) are constructed, \( R(x, U_{n+1}) \geq R(C) \) for each \( x \in U_n \). It follows that \( R(C_1) \geq R(C) \) and the proof is complete.

For \( 0 < \mu < \frac{1}{2} \) and \( p > 2 \), denote by \( x(\mu) \) the unique solution of the equation

\[
\lambda x^{\rho-1} - \mu - (\lambda x - \mu)^{\rho-1} = 0
\]

in the interval \( \mu/\lambda \leq x \leq 1 \). Define \( g(\mu) \), \( 0 \leq \mu \leq 1 \), by

\[
g(\mu) = \frac{\lambda \mu^{-1} + x(\lambda \wedge \mu)^{\rho-1}}{(1 + x(\lambda \wedge \mu)^{\rho-1})^{\rho-1}}
\]

where \( \lambda = 1 - \mu \). We proved in [4] the following inequality in \( L^p \) (\( p > 2 \)):

\[
\| \lambda x + \mu y \|_p + g(\mu) \| x - y \|_p \leq \lambda \| x \|_p + \mu \| x \|_p
\]

and that

\[
\sup_{0 < \mu < 1} \frac{g(\mu)}{\mu} = \frac{1 + \alpha^{\rho-1}}{(1 + \alpha)^{\rho-1}},
\]

where \( \alpha \) is the unique solution of

\[
(p - 2)x^{\rho-1} + (p - 1)x^{\rho-2} - 1 = 0
\]

in the interval \( 0 \leq x \leq 1 \).
THEOREM 2. For $X = L^p$, $p > 2$,

$$N(X) \geqslant \left(1 + \frac{1 + \alpha^{p-1}}{(1 + \alpha)^{p-1}}\right)^{1/p}.$$ 

PROOF. For a closed convex bounded set $C$ in $X$, let $R$ and $D$ be the Chebyshev radius and the diameter of $C$ respectively. Let $z$ be the Chebyshev center of $C$. For $x$, $y$ in $C$ and $0 < \mu \leqslant 1$, we have

$$\|\lambda z + \mu y - x\|_p + g(\mu)\|z - y\|_p \leqslant \lambda \|z - x\|_p + \mu \|y - x\|_p.$$ 

Taking sup over $x$ in $C$ and noting that $R \leqslant \sup\{\|\lambda z + \mu y - x\|: x \in C\}$, we obtain

$$R^p + g(\mu)\|z - y\|_p \leqslant \lambda R^p + \mu \sup\{\|y - x\|_p: x \in C\}.$$ 

It follows, after taking sup over $y$ in $C$, that $(\mu + g(\mu))R^p \leqslant \mu D^p$ and hence

$$\frac{D}{R} \geqslant \left(1 + \sup_{0 < \mu \leqslant 1} \frac{g(\mu)}{\mu}\right)^{1/p} = \left(1 + \frac{1 + \alpha^{p-1}}{(1 + \alpha)^{p-1}}\right)^{1/p}.$$ 

Therefore

$$N(X) \geqslant \left(1 + \frac{1 + \alpha^{p-1}}{(1 + \alpha)^{p-1}}\right)^{1/p}. \quad \square$$

REMARK 1. For $p = 3$ and $4$, we have $\alpha = \sqrt{2} - 1$ and $1/2$ and hence

$$\left(1 + \frac{1 + \alpha^{p-1}}{(1 + \alpha)^{p-1}}\right)^{1/p} = \left(3 - \sqrt{2}\right)^{1/3} \text{ and } (4/3)^{1/4}$$

respectively.

REFERENCES


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