Shorter Notes: The Center of a Convex Set

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THE CENTER OF A CONVEX SET

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Let $X$ be a Banach space and $K$ a weakly compact convex nonvoid subset with normal structure [1]. Brodskii and Mil'man [1] constructed, using transfinite induction, a "center" of $K$ which is fixed by every isometry mapping $K$ onto $K$. In this note, we construct a unique "center" for a weakly compact convex nonvoid subset (not necessarily having normal structure) which is fixed by every affine isometry mapping $K$ into $K$. A similar theorem for weak* compact convex sets is also possible under some additional assumptions.

CONSTRUCTION. Let $K$ be a nonempty weakly compact convex subset of a Banach space. We shall define $C_\alpha$ for all ordinals $\alpha$ by transfinite induction. Set $C_0 = K$. Let $\beta$ be an ordinal and suppose that $C_\alpha$ has been defined for $\alpha < \beta$ in such a way that (i) each $C_\alpha$ is a nonempty closed convex subset of $K$ and (ii) $C_\alpha$, $\alpha < \beta$, is decreasing. If $\beta$ is a limit ordinal, we set $C_\beta = \bigcap_{\alpha<\beta} C_\alpha$. Otherwise, let $\gamma$ be the predecessor of $\beta$ and let

$$s_\beta = \{z \in C_\beta : z = \frac{1}{2}(x + y) \text{ for some } x, y \in C_\gamma \text{ with } \|x - y\| = \frac{1}{2} \text{ diam } C_\gamma\}.$$  

Then we set $C_\beta = \overline{co} s_\beta$. Since $C_\gamma$ is the closed convex hull of its strongly exposed points (see [2]), it is easy to see that if card $C_\gamma > 1$, $C_\beta$ contains no strongly exposed points of $C_\gamma$ and hence is a proper subset of $C_\gamma$. If card $C_\gamma = 1$, $C_\beta = C_\gamma$. It follows that for sufficiently large ordinals $\delta$, $C_\delta$ are identical and consist of exactly one point which we call the center of $K$.

If $X$ is a Banach space such that the dual of every separable subspace of $X$ is separable, and $K$ is a nonempty weak* compact convex subset of $X^*$, then every weak* closed convex nonempty subset of $K$ is the weak* closed convex hull of its weak* strongly exposed points (see [5]-[8]). With appropriate changes, the prior construction applies to this situation; in particular, replacing $C_\beta$ by $\overline{co}^*(S_\beta)$, where $\overline{co}^*$ denotes the weak* closure. Thus $K$ has a unique center.

THEOREM 1. Let $K$ be a nonempty weakly compact convex subset of a Banach space. The center of $K$ is a fixed point of every affine isometry mapping $K$ into $K$.

PROOF. Note that in the construction, each $C_\alpha$ is mapped into itself by every affine isometry of $K$ into $K$.

Received by the editors April 21, 1980.

1980 Mathematics Subject Classification. Primary 46B20; Secondary 47H10.

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THEOREM 2. Let $X$ be a Banach space such that the dual of every separable subspace of $X$ is separable. Let $K$ be a weak* compact convex nonempty subset of $X^*$. The center of $K$ is a fixed point of every weak* continuous affine isometry mapping $K$ into $K$.

PROOF. If $T$ is an affine isometry, then $T(\text{Co } S_\beta) \subseteq \text{Co } S_\beta$. By the weak* continuity, $T(C_\beta) = T(\text{Co } S_\beta) \subseteq C_\beta$.

REMARKS. 1. It also follows from the Ryll-Nardzewski fixed point theorem (see [4]) that the family of affine isometries on $K$ has a common fixed point (which is not necessarily the center). Our approach follows that of Namioka-Asplund [4].

2. The assumption of weak* continuity in Theorem 2 cannot be removed since Example 1 in [3] shows that there are fixed point free affine isometries.

REFERENCES


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