

The MSRI-Korea Publications, Number 1, 1986

Offprints from

Operator Equations and Fixed Point Theorems

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A NOTE ON THE CLARKSON
INEQUALITY AND THE HANNER INEQUALITY

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The Clarkson inequality for $1 < p < 2$ is

$$\|x + y\|^q + \|x - y\|^q \leq 2 (\|x\|^p + \|y\|^p)^{\frac{1}{p-1}}, \quad (1)$$

$\frac{1}{p} + \frac{1}{q} = 1$, $x, y \in L^p$, see e.g., [4]. Clarkson [1] used the inequality to estimate the modulus of convexity of L^p . Hanner obtained the exact formula for the modulus of convexity by applying the following inequality ($1 < p < 2$):

$$|\|x\| - \|y\||^p + (\|x\| + \|y\|)^p \leq \|x - y\|^p + \|x + y\|^p, \quad (2)$$

$x, y \in L^p$, see [3], [2]. In a recent investigation [5], [6] on certain fixed point theorems in L^p spaces, we find it important to know whether the Hanner inequality is stronger than Clarkson's; i.e., whether the latter follows from the former.

Replacing x by $x - y$ and y by $x + y$ in (1), we see that the Clarkson inequality for $1 < p < 2$ is equivalent to the following:

$$2(\|x\|^q + \|y\|^q)^{p-1} \leq \|x - y\|^p + \|x + y\|^p,$$

$\frac{1}{p} + \frac{1}{q} = 1$, $x, y \in L^p$. Thus part of the next theorem answers the above question in the affirmative. We also obtain the reversed inequality for $p > 2$.

Theorem 1. For $x, y \in L^p$, $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, one has the following inequality:

$$\begin{aligned} 2(\|x\|^q + \|y\|^q)^{p-1} &\leq |\|x\| - \|y\||^p + (\|x\| + \|y\|)^p \\ &\leq \|x - y\|^p + \|x + y\|^p \leq 2(\|x\|^p + \|y\|^p) \end{aligned}$$

For $p \geq 2$, the above are valid with the inequality signs reversed. The equality sign holds in the first inequality if and only if $x = 0$ or $y = 0$ or $\|x\| = \|y\|$; in the second inequality if and only if $x = 0$ or $y = 0$

or there is a number $\lambda > 0$ such that $(x(t) - \lambda y(t))(x(t) + \lambda y(t)) = 0$ for almost every t , and in the third inequality if and only if $x(t)y(t) = 0$ for almost every t .

Proof. The last two inequalities were proved in Hanner [3], Clarkson [1]. For the first inequality we divide both sides by $\max(\|x\|^p, \|y\|^p)$ and see that it is equivalent to

$$2(1 + x^q)^{p-1} \leq (1 - x)^p + (1 + x)^p \quad (3)$$

for $0 \leq x \leq 1$. For $1 < p < 2$, this inequality was proved in [4] (p. 225, inequality (3)). For the sake of completeness, we present the proof in [4] here with a slight change.

$$(1 - x)^p + (1 + x)^p = 2(1 + \binom{p}{2}x^2 + \binom{p}{4}x^4 + \dots + \binom{p}{2n}x^{2n} + \dots)$$

$$\text{and } 2(1 + x^q)^{p-1} = 2(1 + \binom{p-1}{1}x^q + \binom{p-1}{2}x^{2q} + \dots + \binom{p-1}{n}x^{nq} + \dots).$$

It suffices to prove that

$$\binom{p-1}{2n-1}x^{(2n-1)q} + \binom{p-1}{2n}x^{2nq} \geq \binom{p}{2n}x^{2n}, \quad n = 1, 2, \dots \quad (4)$$

$$\text{Consider } f(x) = \binom{p}{2n} - \binom{p-1}{2n-1}x^{(2n-1)q-2n} - \binom{p-1}{2n}x^{2n(q-1)}.$$

We have $f(0) = \binom{p}{2n} > 0$ and $f(1) = \binom{p}{2n} - \binom{p-1}{2n-1} - \binom{p-1}{2n} = 0$. Also

$$f'(x) = -\binom{p-1}{2n-1}x^{2nq-q-2n-1}(2nq-q-2n+(q-1)(p-2n)x^q).$$

Since $1 < p < 2 \leq 2n$, we have $q > \frac{2n}{2n-1}$, $2nq-q-2n = -(q-1)(p-2n) > 0$ and $\binom{p-1}{2n-1} > 0$ and so $f'(x) < 0$ for $0 < x < 1$. It follows that $f(x)$ is strictly decreasing and equal zero only when $x = 1$. Thus (3) obtains and equality holds if and only if $x = 0$ or 1 .

For the case $p > 2$, we need to reverse the inequality (3). A naive approach would be to reverse the inequality sign in (4). Unfortunately this approach fails if $\binom{p}{2n} > 0$ and $(2n-1)q > 2n$, e.g., if $p = 7/2$ and $n = 2$, then the reversed inequality (4) fails with $x = 1/2^{10}$. However, we do have the reversed inequality (4) if $2 < p \leq 3$.

For $n = 1$, the inequality

$$\binom{p-1}{1}x^q + \binom{p-1}{2}x^{2q} \geq \binom{p}{2}x^2$$

is valid for any $p > 2$ and equality holds if and only if $x = 0$ or 1 . Indeed if $g(x) = \binom{p-1}{1} + \binom{p-1}{2}x^q - \binom{p}{2}x^{2-q}$, then $g(0) = p-1 > 0$, $g(1) = 0$, $q\binom{p-1}{2} = (2-q)\binom{p}{2} > 0$ and $g'(x) = q\binom{p-1}{2}(x^{q-1} - x^{1-q}) < 0$ for $0 < x < 1$. Hence $g(x) > 0$, $0 < x < 1$.

For $n \geq 2$, and $2 < p \leq 3$, we have $p < 2n$ and so $(2n-1)q - 2n > 0$. Consider $h(x) = \binom{p-1}{2n-1}x^{(2n-1)q-2n} + \binom{p-1}{2n}x^{2nq-2n} - \binom{p}{2n}$. We have $h(0) = -\binom{p}{2n} \geq 0$ and $h(1) = 0$. Also

$$h'(x) = \binom{p-1}{2n-1}x^{(2n-1)q-2n-1}(2nq-q-2n+(q-1)(p-2n)x^q)$$

in which $\binom{p-1}{2n-1} \leq 0$ and $2nq-q-2n = -(q-1)(p-2n) > 0$. Hence $h'(x) \leq 0$ and therefore $h(x) \geq 0$. Thus

$$\binom{p-1}{2n-1}x^{(2n-1)q} + \binom{p-1}{2n}x^{2nq} \geq \binom{p}{2n}x^{2n}.$$

If $p > 3$, we use the following method of reduction of p (cf. [4] p. 223): Consider

$$\begin{aligned} K(x) &= \frac{1}{x^p} (2(1+x^q)^{p-1} - (1-x)^p - (1+x)^p) \\ &= 2\left(\frac{1}{x^q} + 1\right)^{p-1} - \left(\frac{1}{x} - 1\right)^p - \left(\frac{1}{x} + 1\right)^p. \end{aligned}$$

We have $K(1) = 0$ and

$$K'(x) = -\frac{p}{x^{p+1}} (2(1+x^q)^{p-2} - (1-x)^{p-1} - (1+x)^{p-1}).$$

Let $p' = p-1$ and q' be such that $\frac{1}{p'} + \frac{1}{q'} = 1$. Then $q' = \frac{p-1}{p-2} > q = \frac{p}{p-1}$ and $x^{q'} < x^q$ for $0 < x < 1$. Thus if we can prove that

$$2(1+x^{q'})^{p'-1} - (1-x)^{p'} - (1+x)^{p'} > 0$$

for $0 < x < 1$, then $K'(x) < 0$ and $K(x)$ is strictly decreasing and so $K(x) > 0$, i.e.,

$$2(1+x^q)^{p-1} - (1-x)^p - (1+x)^p > 0$$

for $0 < x < 1$. So we see that we could reduce p by one. Successively reducing p to between 2 and 3 will then prove the theorem.

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