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NOTE

A Fixed Point Theorem for Weakly Inward Multivalued Contractions

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Let D be a closed nonempty subset of a Banach space X and $T\colon D \to 2^X \setminus \{\phi\}$ a multivalued contraction with closed values, i.e., each Tx is a nonempty closed subset of X and there exists $0 \le k < 1$ such that

$$H(Tx, Ty) \le k\|x - y\|, \qquad x, y \in D,$$

where H denotes the Hausdorff metric

$$H(A, B) = \max\left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}.$$

($d(a, A)$ will denote $\inf\{\|a - b\| : b \in A\}$, the distance from a to A, throughout this paper.)

T is said to be weakly inward if

$$Tx \subset x + \overline{\{\lambda(z - x) : z \in D, \lambda \ge 1\}}. \tag{1}$$

The objective of this paper is to prove that T has a fixed point. The result was previously proved by Martines-Yanez [5] for single-valued T, by Yi and Zhao [7] for compact-valued T, and by Xu [6] for T satisfying the condition that each x in D has a nearest point in Tx. One significant difference between our proof and the ones in these papers is that Caristi's theorem is not used here. In fact, the author had attempted unsuccessfully to use Caristi's theorem.

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Note that in Deimling [1] and elsewhere (e.g., Martin [4]), a mapping $T : D \to 2^X \setminus \{\phi\}$ is called *weakly inward* if

$$Tx \subset x + S_D(x), \tag{2}$$

where

$$S_D(x) = \left\{ y \in X : \liminf_{\lambda \to 0+} \lambda^{-1} d(x + \lambda y, D) = 0 \right\}.$$

One can easily show that the set in (1) is a superset of $x + S_D(x)$ in (2) (see the remark below). For closed convex sets, the two sets are identical. Simple examples (see Example 11.1 of [1]) show that the former can be much larger so Condition (1) is more general than Condition (2). Our result extends Theorem 11.4 in [1] (where T is assumed to satisfy Condition (2), and each x is assumed to have a nearest point in Tx) and completely solves Problem 7 from Section 11 of [1]. Since we do not assume that each x has a nearest point in Tx, our result is new even when the mapping satisfies the stronger condition (2) or when D is convex.

We refer the reader to the book [1] or the article by Downing and Kirk [2] for some relevant historical accounts.

THEOREM 1. *Let D be a nonempty closed subset of a Banach space X and T: $D \to 2^X \setminus \{\phi\}$ a multivalued contraction with closed values. Assume that T is weakly inward. Then T has a fixed point, i.e., there exists $x \in D$ such that $x \in Tx$.*

Proof. Let $0 \leq k < 1$ be the contraction constant of T. Let $k < l < 1$ and $0 < \epsilon < 1$ be such that $b = (1 - \epsilon)/(1 + \epsilon) - l > 0$.

Assume on the contrary that T does not have fixed points.

Let $z_0 \in D$ be arbitrary and y_0 be an arbitrary point of Tz_0.

We will follow a transfinite induction as used in [3]. Let Ω be the first uncountable ordinal. Let γ be an ordinal $< \Omega$. Suppose y_α, z_α have been defined for all $\alpha < \gamma$ such that

(i) $y_\alpha \in Tz_\alpha$ for $\alpha < \gamma$

(ii) $z_\alpha \neq z_{\alpha+1}$ for $\alpha < \alpha + 1 < \gamma$

(iii) $b \max\{\|z_\beta - z_\alpha\|, \frac{1}{l}\|y_\beta - y_\alpha\|\} \leq \|y_\alpha - z_\alpha\| - \|y_\beta - z_\beta\|$ for $\alpha < \beta < \gamma$.

We proceed to define y_γ and z_γ so that (i)–(iii) remain valid for all $\alpha, \beta < \gamma + 1$.

Case (i). γ has a predecssor $\gamma - 1$.

Since $y_{\gamma-1} \in Tz_{\gamma-1}$ and T is fixed-point free, we have $\|y_{\gamma-1} - z_{\gamma-1}\| > 0$. By the weak inwardness of T there exist $z_\gamma \in D$, $\lambda_\gamma \geq 1$, such that

$$\left\| y_{\gamma-1} - \left(z_{\gamma-1} + \lambda_\gamma (z_\gamma - z_{\gamma-1}) \right) \right\| \leq \epsilon \| y_{\gamma-1} - z_{\gamma-1} \|.$$

This, together with $0 < \epsilon < 1$, implies that $z_{\gamma-1} \neq z_\gamma$ and

$$\|z_\gamma - z_{\gamma-1}\| \le (1+\epsilon)\,\mu_\gamma \|y_{\gamma-1} - z_{\gamma-1}\|$$
$$\|z_\gamma - x_\gamma\| \le \epsilon\mu_\gamma \|y_{\gamma-1} - z_{\gamma-1}\|,$$

where $\mu_\gamma = 1/\lambda_\gamma$ and $x_\gamma = \mu_\gamma y_{\gamma-1} + (1-\mu_\gamma) z_{\gamma-1}$.

Since $H(Tz_\gamma, Tz_{\gamma-1}) \le k\|z_\gamma - z_{\gamma-1}\|$, there exists a $y_\gamma \in Tz_\gamma$ such that

$$\|y_\gamma - y_{\gamma-1}\| \le l\|z_\gamma - z_{\gamma-1}\|. \tag{3}$$

Thus

$$\begin{aligned}\|y_\gamma - z_\gamma\| &\le \|y_\gamma - y_{\gamma-1}\| + \|y_{\gamma-1} - x_\gamma\| + \|x_\gamma - z_\gamma\| \\ &\le l\|z_\gamma - z_{\gamma-1}\| + \big(1-\mu_\gamma\big)\|y_{\gamma-1} - z_{\gamma-1}\| + \epsilon\mu_\gamma\|y_{\gamma-1} - z_{\gamma-1}\| \\ &\le l\|z_\gamma - z_{\gamma-1}\| + \|y_{\gamma-1} - z_{\gamma-1}\| + \frac{\epsilon - 1}{1+\epsilon}\|z_\gamma - z_{\gamma-1}\|,\end{aligned}$$

from which it follows that

$$b\|z_\gamma - z_{\gamma-1}\| \le \|y_{\gamma-1} - z_{\gamma-1}\| - \|y_\gamma - z_\gamma\|$$

and, from (3),

$$\frac{b}{l}\|y_\gamma - y_{\gamma-1}\| \le \|y_{\gamma-1} - z_{\gamma-1}\| - \|y_\gamma - z_\gamma\|.$$

For any $\alpha < \gamma - 1$,

$$b\|z_\alpha - z_{\gamma-1}\| \le \|y_\alpha - z_\alpha\| - \|y_{\gamma-1} - z_{\gamma-1}\|$$

by (iii). So

$$b\|z_\gamma - z_\alpha\| \le b\big(\|z_\gamma - z_{\gamma-1}\| + \|z_{\gamma-1} - z_\alpha\|\big) \le \|y_\alpha - z_\alpha\| - \|y_\gamma - z_\gamma\|.$$

Similarly,

$$\frac{b}{l}\|y_\gamma - y_\alpha\| \le \|y_\alpha - z_\alpha\| - \|y_\gamma - z_\gamma\|.$$

So (i)–(iii) are valid for $\alpha, \beta < \gamma + 1$.

Case (ii). γ is a limit ordinal.

Let $\{\gamma_n\}$ be a strictly increasing sequence of ordinals with limit γ. Let $r_n = \|y_{\gamma_n} - z_{\gamma_n}\|$. Condition (iii) implies that $\{r_n\}$ is decreasing and hence convergent. Condition (iii) then implies that $\{z_{\gamma_n}\}$ and $\{y_{\gamma_n}\}$ are Cauchy sequences and hence convergent. Let z_γ and y_γ be their respective limits. Since $y_{\gamma_n} \in Tz_{\gamma_n}$ there exists $w_n \in Tz_\gamma$ such that $\|w_n - y_{\gamma_n}\| \leq l\|z_{\gamma_n} - z_\gamma\|$. Thus $w_n - y_{\gamma_n} \to 0$. Since $\lim_n y_{\gamma_n} = y_\gamma$ we have $\lim_n w_n = y_\gamma$. Thus $y_\gamma \in Tz_\gamma$ since Tz_γ is closed.

For any $\alpha < \gamma$, we have $\gamma_n > \alpha$ for sufficiently large n, so

$$b\|z_{\gamma_n} - z_\alpha\| \leq \|y_\alpha - z_\alpha\| - \|y_{\gamma_n} - z_{\gamma_n}\|,$$

and upon taking the limit,

$$b\|z_\gamma - z_\alpha\| \leq \|y_\alpha - z_\alpha\| - \|y_\gamma - z_\gamma\|.$$

Similarly,

$$\frac{b}{l}\|y_\gamma - y_\alpha\| \leq \|y_\alpha - z_\alpha\| - \|y_\gamma - z_\gamma\|.$$

Hence (i) and (iii) remain valid for all $\alpha, \beta < \gamma + 1$. If $\alpha < \alpha + 1 < \gamma + 1$, then $\alpha < \gamma$. Since γ is a limit ordinal, $\alpha + 1 < \gamma$. So (ii) is also valid for $\alpha < \alpha + 1 < \gamma + 1$.

By transfinite induction, y_α, z_α, for $\alpha < \Omega$ satisfying (i)–(iii), have been defined. Let $s_\alpha = \|y_\alpha - z_\alpha\|$. Since $\{s_\alpha\}_{\alpha < \Omega}$ is decreasing and bounded below by 0, it must be eventually constant. If $\gamma < \Omega$ is such that $s_\alpha = s_\beta$ for all $\alpha, \beta \geq \gamma$, then by (iii) $z_{\gamma+1} = z_\gamma$, contradicting (ii).

Hence T has a fixed point.

Remark 1. We show that $S_D(x)$ in (2) is a subset of $\overline{\{\lambda(z - x) : z \in D, \lambda \geq 1\}}$. If $y \in S_D(x)$, then for every $\epsilon > 0$ there exist $\lambda, 0 < \lambda < 1$, and $z \in D$ such that $\|x + \lambda y - z\| < \lambda\epsilon$. Dividing by λ gives $\|y - \mu(z - x)\| < \epsilon$, where $\mu = \frac{1}{\lambda} > 1$. So $y \in \overline{\{\lambda(z - x) : z \in D, \lambda \geq 1\}}$.

Remark 2. Theorem 1 can be easily generalized to metrically convex metric spaces. In that case condition (1) is replaced with

$$Tx \subset \overline{\{y \in X : \exists z \in D \text{ s.t. } z \in [x, y]\}} \tag{4}$$

and the proof carries over in the same manner.

Thus we have

THEOREM 2. *Let D be a nonempty closed subset of a complete metrically convex metric space X and $T : D \to 2^X \setminus \{\phi\}$ be a multivalued contraction with closed values. Assume that T is weakly inward in the sense that it satisfies* (4). *Then T has a fixed point, i.e., there exists $x \in D$ such that $x \in Tx$.*

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