Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE @ DIRECT

J. Math. Anal. Appl. 282 (2003) 1–7

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Fixed points of isometries on weakly compact convex sets

Teck-Cheong Lim,[a] Pei-Kee Lin,[b] C. Petalas,[c] and T. Vidalis [c,*]

[a] *Department of Mathematical Sciences, George Mason University, 4400, University Drive, Fairfax, VA 22030, USA*
[b] *Department of Mathematical Sciences, University of Memphis, Memphis, TN, USA*
[c] *Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece*

Received 22 October 2001

Submitted by W.A. Kirk

Abstract

In this paper, we prove that every isometry from a nonempty weakly compact convex set K into itself fixes a point in the Chebyshev center of K, provided K satisfies the hereditary fixed point property for isometries. In particular, all isometries from a nonempty bounded closed convex subset of a uniformly convex Banach space into itself have the Chebyshev center as a common fixed point.

Keywords: Weakly compact convex set; Isometry; Fixed point; Uniformly convex Banach space

1. Introduction

Let X be a Banach space and S a bounded subset of X. For $x \in X$, we write

$$R(x,S) = \sup\big\{\|x-y\|:\ y \in S\big\},$$

and we denote the diameter of S by $\operatorname{diam} S$, i.e., $\operatorname{diam} S = \sup\{\|y-z\|:\ y,z \in S\}$. For $r \geqslant 0$, $B(x,r)$ denotes the closed ball with radius r centered at x. $\overline{\mathrm{co}}(S)$ denotes the closure of the convex hull of S.

* Corresponding author.
E-mail addresses: tlim@gmu.edu (T.-C. Lim), pklin@msci.memphis.edu (P.-K. Lin), cpetalas@cc.uoi.gr (C. Petalas), tvidalis@cc.uoi.gr (T. Vidalis).

0022-247X/03/$ – see front matter
doi:10.1016/S0022-247X(03)00398-6

Let K be a weakly compact convex nonempty subset of X. Let $R_K = \inf\{R(x, K)\colon x \in K\}$. The set $\{x \in K\colon R(x, K) = R_K\}$ and the number R_K are called, respectively, the *Chebyshev center* and the *Chebyshev radius* of K. The subset K of X is said to have *normal structure* if every convex subset C of K with more than one point has a nondiametral point, i.e., a point $x_0 \in C$, such that

$$R(x_0, C) < \operatorname{diam} C.$$

A mapping $T : K \to K$ is called *isometry* if $\|Tx - Ty\| = \|x - y\|$ for all x, y in K.

The following proposition is known; we give its proof for completeness.

Proposition 1. *The Chebyshev center C of K is nonempty. If $T : K \to K$ is an isometry from K onto K, then $T(C) = C$.*

Proof. We have

$$C = \bigl\{x \in K\colon R(x, K) = R_K\bigr\} = \bigcap_{n\in\mathbb{N}} C_n,$$

where each $C_n = \{x \in K\colon R(x, K) \leqslant R_K + 1/n\}$ is nonempty closed convex. Since K is weakly compact and $\{C_n\}$ is decreasing, $C \neq \emptyset$.

Let $x \in C$. Since T is a surjective isometry, one has

$$R(Tx, K) = R(Tx, TK) = R(x, K) = R_K$$

and hence $Tx \in C$. So $T(C) \subset C$.

On the other hand, applying the above argument to T^{-1}, we get that $x \in C$ implies $y = T^{-1}x \in C$. So $x = Ty \in T(C)$, showing that $C \subset T(C)$. □

Question 1. In Proposition 1, if T is not surjective, does one still have $T(C) \subset C$?

We will see later that the answer to the above question is positive under certain assumptions.

2. Isometries in uniformly convex spaces

Throughout this section we denote by X a uniformly convex Banach space and by K a nonempty bounded closed and convex subset of X. Since every uniformly convex Banach space is reflexive, the set K is weakly compact.

An equivalent definition of uniform convexity is: for every $\epsilon > 0$ and $r > 0$, there exists $\delta(\epsilon) > 0$ such that whenever $x, y, z \in X$, $\|x - z\| \leqslant r$, $\|y - z\| \leqslant r$ and $\|x - y\| \geqslant \epsilon$, one has

$$\left\| \frac{x + y}{2} - z \right\| \leqslant r - \delta(\epsilon).$$

Lemma 1. *Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space X. The Chebyshev center C of K is a singleton, say k_0.*

Proof. That $C \neq \emptyset$ follows from Proposition 1.

Suppose on the contrary that $x_1, x_2 \in C$, and let $\|x_1 - x_2\| = \epsilon > 0$. Let $r = R_K$. For every $z \in K$, we have $\|x_1 - z\| \leqslant r$ and $\|x_2 - z\| \leqslant r$. By the definition of uniform convexity, there exists a $\delta(\epsilon) > 0$ such that

$$\left\| \frac{x_1 + x_2}{2} - z \right\| < r - \delta(\epsilon). \tag{1}$$

By the definition of R_K, there exists a z in K such that

$$\left\| \frac{x_1 + x_2}{2} - z \right\| > r - \delta(\epsilon), \tag{2}$$

which contradicts (1). □

Proposition 2. *Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space X. If T is an isometry, $T : K \to K$, then $T(K)$ is also nonempty bounded closed convex.*

Proof. It is well known that any isometry in a uniformly convex Banach space is affine [2]. So the convexity of $T(K)$ is obvious.

If $\{Tx_n\}$ is a sequence in $T(K)$, which converges to y, then $\{Tx_n\}$ is a Cauchy sequence. By isometry $\{x_n\}$ is also Cauchy and so converges to a point x of the complete set K. Therefore $y = Tx \in T(K)$ and $T(K)$ is closed. □

Lemma 2. *The Chebyshev center k of K is a fixed point for all isometries from K onto K.*

Proof. By Lemma 1, the Chebyshev center is unique. By Proposition 1, it is a fixed point of every isometry from K onto K. □

Lemma 3. *Let $T : K \to K$ be an isometry. Then there exists a k in K such that $Tk = k$.*

Proof. We have $T^2(K) \subset T(K) \subset K$ and inductively $T^{n+1}(K) \subset T^n(K)$. Consider the set $L := \bigcap\{T^n(K)\colon n \in \mathbb{N}\}$. Since $T^n(K)$ is weakly compact for every $n \in \mathbb{N}$, it follows that $L \neq \emptyset$. Using the fact that T is one-to-one, it is not difficult to show that $T(L) = L$, and so our conclusion follows from Lemma 2. □

Lemma 4. *Let $T : K \to K$ be an isometry and consider the set $S = \overline{\mathrm{co}}\{T^n x_0\colon n \in \mathbb{N}\}$, where x_0 is an arbitrary point of K. Then T has a fixed point in S.*

Proof. Obviously since T is affine and continuous, $T(S) \subset S$, and the proof follows from Lemma 3. □

Lemma 5. *Let K be as in Lemma* 1 *and $T : K \to K$ an isometry. Then the Chebyshev radius $R_{T(K)}$ of $T(K)$ is equal to R_K and the Chebyshev center of $T(K)$ coincides with the point Tk_0.*

Proof. For the point $Tx_0 \in T(K)$ we have

$$\|Tk_0 - Tx\| = \|k_0 - x\| \leqslant R_K, \quad \forall Tx \in T(K), \tag{3}$$

which implies that $R_{T(K)} \leqslant R_K$.

Suppose for a contradiction that $R_{T(K)} < R_K$ and let k_0' be the Chebyshev center of $T(K)$. Then there is a point $y \in K$ such that $Ty = k_0'$. Now for every $x \in K$ we have

$$\|x - y\| = \|Tx - Ty\| \leqslant R_{T(K)} < R_K,$$

which contradicts the definition of R_K.

Hence $R_{T(K)} = R_K$ and (3) implies that $T(k_0)$ is the Chebyshev center of $T(K)$. □

Now we can prove our result:

Theorem 1. *Let K be a nonempty bounded closed and convex subset of a uniformly convex Banach space X. Then the Chebyshev center k_0 of K is a fixed point for every isometry $T: K \to K$.*

Proof. Suppose on the contrary that $Tk_0 \neq k_0$ and consider the set $S = \overline{\text{co}}\{T^n k_0\colon n \in \mathbb{N}\}$. From Lemma 4 we have that there is a point $z \in S$ with $Tz = z$. Obviously $z \in T^n(K)$, $\forall n \in \mathbb{N}$, and because z is not the Chebyshev center of K there exists a point $e \in K$ such that

$$\|z - e\| = R_K + t \tag{4}$$

for some $t > 0$. Since $z \in S$, it follows that there exist nondecreasing sequence $n_1, n_2, \ldots, n_k \in \mathbb{N}$ with

$$\left\| \sum_{i=1}^{k} \lambda_i T^{n_i} k_0 - z \right\| < \frac{t}{2}, \tag{5}$$

where $\sum_{i=1}^{k} \lambda_i = 1$, $\lambda_i > 0$, $\forall i = 1, 2, \ldots, k$. From (4) and (5) it follows that

$$\begin{aligned} \left\| \sum_{i=1}^{k} \lambda_i T^{n_i} k_0 - T^{n_k} e \right\| &\geqslant \|z - T^{n_k} e\| - \left\| z - \sum_{i=1}^{k} \lambda_i T^{n_i} k_0 \right\| \\ &\geqslant R_K + t - \frac{t}{2} = R_K + \frac{t}{2}. \end{aligned} \tag{6}$$

On the other hand, since $T^{n_i} k_0, T^{n_k} e \in T^{n_i}(K)$, $\forall i = 1, 2, \ldots, k$, Lemma 5 implies that $\|T^{n_i} k_0 - T^{n_k} e\| \leqslant R_K$, and we have

$$\left\| \sum_{i=1}^{k} \lambda_i T^{n_i} k_0 - T^{n_k} e \right\| \leqslant \sum_{i=1}^{k} \lambda_i \|T^{n_i} k_0 - T^{n_k} e\| \leqslant R_K. \tag{7}$$

Now (7) contradicts (6) and we obtain the result. □

3. Isometries on weakly compact convex sets

Let K be a weakly compact convex subset of a Banach space. Let $\{B_\alpha\colon \alpha \in \Lambda\}$ be a decreasing net of nonempty subsets of K and for each $x \in K$ let

$$r\big(x, \{B_\alpha\colon \alpha \in \Lambda\}\big) = \lim_\alpha R(x, B_\alpha)$$

and

$$r = r\big(K, \{B_\alpha\colon \alpha \in \Lambda\}\big) = \inf\big\{r\big(x, \{B_\alpha\colon \alpha \in \Lambda\}\big)\colon x \in K\big\}.$$

The number r and the set $\{x \in K\colon r(x, \{B_\alpha\colon \alpha \in \Lambda\}) = r\}$, which is nonempty closed convex, are called, respectively, the *asymptotic radius* and *asymptotic center* of $\{B_\alpha\colon \alpha \in \Lambda\}$ with respect to (w.r.t.) K.

The following can be found in [6].

Proposition 3. *If $T : K \to K$ is nonexpansive, i.e., $\|Tx - Ty\| \leqslant \|x - y\|$, $\forall x, y \in K$, then the asymptotic center of the sequence of sets $T^n(K)$ is T-invariant.*

Proof. Let C be the asymptotic center of $\{T^n(K)\}$ w.r.t. K and r be its asymptotic radius. Let $x \in C$. For each $y \in K$ we have

$$\|T^n y - Tx\| \leqslant \|T^{n-1} y - x\|$$

which implies

$$R\big(Tx, T^n(K)\big) \leqslant R\big(x, T^{n-1}(K)\big)$$

and hence

$$r\big(Tx, \big\{T^n(K)\big\}\big) \leqslant r\big(x, \big\{T^n(K)\big\}\big).$$

Therefore $Tx \in C$. □

We say a weakly compact convex nonempty subset K of a Banach space have *fixed point property* (f.p.p.) if every isometry of K into K has a fixed point. K is said to have *hereditary f.p.p.* if every closed convex nonempty subset of K has the f.p.p. It is well known (see Kirk [5]) that every weakly compact convex nonempty subset of a Banach space with normal structure has the hereditary f.p.p. Also from a result of Maurey (see [3, Theorem F]), every closed convex bounded nonempty subset of a superreflexive space has hereditary f.p.p. The following theorem is more general than Theorem 1. Since the proofs in Section 2 are essentially self-contained, we think that Theorem 1 warrants separate treatment.

Theorem 2. *Let K be a weakly compact convex nonempty subset of a Banach space, and assume that K has the hereditary f.p.p. Let $T : K \to K$ be an isometry. Then T has a fixed point in the Chebyshev center of K.*

Proof. Let C and r be the asymptotic center and the asymptotic radius, respectively, of the sequence $T^n(K)$ w.r.t. K. Since C is T-invariant, T has a fixed point c in C. Since T is an isometry and $T(c) = c$, we have

$$R\bigl(c, T^n(K)\bigr) = R\bigl(Tc, T^n(K)\bigr) = R\bigl(c, T^{n-1}(K)\bigr), \quad n = 1, 2, \dots,$$

from which it follows that $r = R(c, K)$. For any $x \in K$, we have

$$R(x, K) \geqslant \lim_n R\bigl(x, T^n(K)\bigr) \geqslant r = R(c, K).$$

Thus c is in the Chebyshev center of K. □

The following simple example shows that the isometry T in the theorem above need not fix all points in the Chebyshev center.

Example 1. Let X be $\mathbb{R}^2$ with the sup norm. Let K be the right half of the unit ball of X. The Chebyshev center of K is $\{(x, 0)\colon 0 \leqslant x \leqslant 1\}$ and the isometry $T(x, y) = (1 - x, y)$ fixes only $(1/2, 0)$ in the Chebyshev center.

Corollary 1. *Let K be a weakly compact convex nonempty subset of a Banach space, and assume that K has normal structure. Let $T: K \to K$ be an isometry. Then T has a fixed point in the Chebyshev center of K.*

Corollary 2. *Let K be a closed convex bounded nonempty subset of a superreflexive Banach space, and $T: K \to K$ be an isometry. Then T has a fixed point in the Chebyshev center of K.*

Corollary 3. *Let K be a weakly compact convex nonempty subset of a Banach space whose norm is uniformly convex in every direction. Then every isometry $T: K \to K$ has the Chebyshev center of K as a fixed point.*

Proof. The Chebyshev center of K is unique [4]. □

Problem 1. Let K be a weakly compact convex nonempty subset of a Banach space, and assume that K has normal structure. Brodskii–Milman [1] proved that there is a point that is fixed by every surjective isometry of K. Does K have a point that is fixed by every isometry from K into K?

References

[1] M.S. Brodskii, D.P. Milman, On the center of a convex set, Dokl. Akad. Nauk SSSR 59 (1948) 837–840, in Russian.

[2] M. Edelstein, A theorem on fixed points under isometries, Amer. Math. Monthly 70 (1963) 298–300.

[3] J. Elton, P.K. Lin, E. Odell, S. Szarek, Remarks on the fixed point problem for nonexpansive maps, in: Fixed Points and Nonexpansive Mappings (Cincinnati, Ohio, 1982), in: Contemp. Math., Vol. 18, American Mathematical Society, Providence, RI, 1983, pp. 87–120.

[4] A.L. Garkavi, The best possible net and the best possible cross-section of a set in a normed space, Izv. Akad. Nauk SSSR Ser. Mat. 26 (1962) 87–106;
A.L. Garkavi, Amer. Math. Soc. Transl. Ser. 2 39 (1964) 111–132.

[5] W.A. Kirk, A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly 72 (1965) 1004–1006.

[6] T.C. Lim, On asymptotic centers and fixed points of nonexpansive mappings, Canad. J. Math. 32 (1980) 421–430.