On minimal (maximal) common fixed points of a commuting family of decreasing (increasing) maps

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Abstract. We prove that in a complete partially ordered set, every commutative family of decreasing maps has a minimal common fixed point.

Let $(X, \leq)$ be a partially ordered set. We call X *complete* if every linearly ordered subset of X has a greatest lower bound in X.
A mapping $f : X \to X$ is called an *decreasing* (resp. *increasing*) if $f(x) \leq x$ (resp. $f(x) \geq x$) for every $x \in X$. A family $\mathcal{F}$ of mappings of X into X is called *commutative* if $f \circ g = g \circ f$ for every $f, g \in \mathcal{F}$, where $\circ$ denotes the composition of maps.

A subset S of $(X, \leq)$ is a directed set if every two, and hence finitely many, elements of S has a *lower* bound in S. For a set S, $|S|$ denotes the cardinality of S.

We shall use the following known fact whose proof can be found in [2]:

Proposition 1 *Let $(X, \leq)$ be a complete partially ordered set. Then every directed subset of X has a greatest lower bound in X.*

Now we prove the following theorem:

Theorem 1 *Let $(X, \leq)$ be a nonempty complete partially ordered set. Let $\mathcal{F}$ be a commutative family of decreasing maps of X into X. Then $\mathcal{F}$ has a minimal common fixed point.*

Proof. First we prove that $\mathcal{F}$ has a common fixed point. Let x_0 be an arbitrary element of X. Let $\mathcal{S}$ be the (commutative) semigroup generated by $\mathcal{F}$. It is clear that each member of $\mathcal{S}$ is also decreasing. Let

$$D = \mathcal{S}(x_0) = \{s(x_0) : s \in \mathcal{S}\}.$$

D is a directed subset of X, since $s(x_0) \leq x_0$ and $t(x_0) \leq x_0$ imply $s \circ t(x_0) = t \circ s(x_0) \leq t(x_0), s(x_0)$ for any $s, t \in \mathcal{S}$. By completeness assumption and Proposition 1, $\mathcal{S}(x_0)$ has a greatest lower bound, which we denote by $\bigwedge \mathcal{S}(x_0)$.
Define $x_1 = \bigwedge \mathcal{S}(x_0)$. Suppose x_α have been defined for $\alpha < \gamma$ such that $x_\alpha \leq x_\beta$ for $\alpha < \beta < \gamma$. If γ has a predecessor $\gamma - 1$, we define $x_\gamma = \bigwedge \mathcal{S}(x_{\gamma-1})$ (the greatest lower bound exists since $\mathcal{S}(x_{\gamma-1})$ is a directed set as in the case for x_0). If γ is a limit ordinal, we define $x_\gamma = \bigwedge\{x_\alpha : \alpha < \gamma\}$. By transfinite induction, x_α is defined for all ordinal α and satisfies $x_\alpha \leq x_\beta$ for $\alpha < \beta$.
Let κ be an ordinal with $|\kappa| > |X|$. Then $x_\alpha, \alpha < \kappa$, cannot be all distinct, so there exists α_1, β with $\alpha_1 < \beta$ such that $x_{\alpha_1} = x_\beta$. This implies that $x_{\alpha_1} = x_{\alpha_1+1}$. It follows from the definition of x_{α_1+1} and the decreasingness of members of $\mathcal{S}$ that

$$s(x_{\alpha_1}) = x_{\alpha_1}$$

for all $s \in \mathcal{S}$ i.e. x_{α_1} is a common fixed point of $\mathcal{S}$ and hence of $\mathcal{F}$.
Now we let K be the set of common fixed points of $\mathcal{F}$, which is nonempty by the above proof. Let C be a chain in K. By completeness, C has a (greatest) lower bound c_0 in X which may not be a common fixed point of $\mathcal{S}$. The set $L = \{x \in X : x \leq c_0\}$ with the order induced by the order of X is complete and is S-invariant, i.e. $s(x) \in L \forall x \in L$. So by the above proof, there is a common fixed point of S in L, which is $\leq$ every member of C. Hence by Zorn's lemma, the set K has a minimal element. This completes the proof.

Corollary 1 *Let $(X, \leq)$ be a nonempty complete partially ordered set and f be a decreasing map from X into X. Then f has a minimal fixed point.*

By considering the dual of X, we obtain the following theorem and its corollary.

Theorem 2 *Let $(X, \leq)$ be a nonempty partially ordered set such that every chain in X has a least upper bound. Let $\mathcal{F}$ be a commutative family of increasing maps of X into X. Then $\mathcal{F}$ has a maximal common fixed point.*

Corollary 2 *Let $(X, \leq)$ be a nonempty partially ordered set such that every chain in X has a least upper bound, and f be an increasing map from X into X. Then f has a maximal fixed point.*

Remark
1. Note that in the last corollary, f may not have a minimal fixed point as erroneously stated in [1] (p.188, Theorem 8.23). For example, let $X =$

$\{-\infty, \cdots, -2, -1, 0\}$ with the usual ordering, and let $f(-\infty) = 0, f(x) = x \forall x \neq -\infty$. Then f is clearly increasing and has no minimal fixed point.

2. It is clear from the proofs that all results above remain valid if one uses well ordered sets instead of chains in definitions of completeness.

References

[1] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, 2nd ed. 2002, Cambridge University Press.

[2] T. C. Lim, *On the largest common fixed point of a commuting family of isotone maps*, Discrete and Continuous Dynamical Systems. Series A (2005),suppl., 621-623.