On Characterizations of Meir-Keeler Contractive Maps

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Abstract

Let (X,d) be a metric space. A map $T : X \to X$ is called Meir-Keeler contractive if

$$\forall \epsilon > 0 \ \exists \delta > 0 \text{ such that } \epsilon \le d(x,y) < \epsilon + \delta \Rightarrow d(Tx,Ty) < \epsilon$$

We introduce "L-functions" and characterize Meir-Keeler contractive maps as maps that satisfy $d(Tx,Ty) < \phi(d(x,y))$ for some L-function ϕ. This characterization makes it easy to compare such maps with those satisfying the Boyd-Wong's condition.

Keywords: Meir-Keeler, Boyd-Wong, contractive maps, fixed point theorems.

Let (X,d) be a complete metric space and $T : X \to X$ a map. Suppose there exists a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\phi(0) = 0, \phi(s) < s$ for $s > 0$ and that ϕ is right upper semicontinuous such that

$$d(Tx,Ty) \le \phi(d(x,y)) \ \forall x,y \in X.$$

Boyd-Wong [1] showed that T has a unique fixed point.
Later, Meir-Keeler [2] extended Boyd-Wong's result to mappings satisfying the following more general condition:

$$\forall \epsilon > 0 \ \exists \delta > 0 \text{ such that } \epsilon \le d(x,y) < \epsilon + \delta \Rightarrow d(Tx,Ty) < \epsilon \tag{1}$$

In this paper, we characterize condition (1) in terms of a ϕ function as in Boyd-Wong's theorem. This is obviously desirable since then one can easily see how much more general is Meir-Keeler's result than Boyd-Wong's. A characterization was given earlier by Wong [4], but it was in terms of a function δ imposed on $d(Tx, Ty)$ rather than $d(x, y)$.

Definition 1 Let ζ be a nondecreasing function from $[0, \infty)$ to $[0, \infty]$. The *pseudo-inverse* of ζ is the function defined by:

$$\psi(t) = \sup\{s : \zeta(s) \leq t\}$$

for $t \in [0, \infty)$.

Proposition 1 Let ζ be a nondecreasing function from $[0, \infty)$ to $[0, \infty]$ and ψ its pseudo-inverse. Then:

1. $\psi(t) \in [0, \infty]$ $\forall t \in [0, \infty)$.
2. ψ is nondecreasing and right continuous.
3. At every point s where ζ is discontinuous, $\psi(t) = s$ $\forall t \in [\zeta(s-), \zeta(s+))$.

Proof.
(1) is obvious. Note that $\psi(t) = \infty$ if and only if $\zeta(s) \leq t$ for all $s \in [0, \infty)$. That ψ in (2) is nondecreasing is obvious. To prove the right continuity, let $t_1 > t_2 > \cdots > t_n > \cdots$ and $\lim_n t_n = t$. Suppose that for large n $\psi(t_n) = c$ for some constant c. By the monotoncity of ζ, we have $\zeta(s) \leq t_n$ $\forall s < c$ for large n; thus, $\zeta(s) \leq t$ $\forall s < c$. This implies that $\psi(t) \geq c$. But $\psi(t) \leq \psi(t_n) = c$. So $\psi(t) = c$ and ψ is right continuous at t in this case. So we may assume that all $\psi(t_n)$ are distinct and hence strictly decreasing. For each n choose s_n such that $\zeta(s_n) \leq t_n$ and $\psi(t_{n+1}) < s_n \leq \psi(t_n)$. Then $s_n \to s = \psi(t+)$ and $\zeta(s) \leq \lim_n \zeta(s_n) \leq \lim_n t_n = t$. By the definition of $\psi(t)$, we then have $\psi(t) \geq s = \psi(t+)$. But $\psi(t) \leq \psi(t+)$ by monotoncity. Hence $\psi(t) = \psi(t+)$, proving the right continuity.
Let $r \in [\zeta(s-), \zeta(s+))$. For any $t > s$, one has $\zeta(t) \geq \zeta(s+) > r$ and hence $\psi(r) \leq t$. Since $t > s$ is arbitrary, we get $\psi(r) \leq s$. On the other hand, for any $t < s$, one has $\zeta(t) \leq \zeta(s-) \leq r$ which implies $\psi(r) \geq t$; and since $t < s$ is arbitrary, $\psi(r) \geq s$. This proves that $\psi(r) = s$ $\forall r \in [\zeta(s-), \zeta(s+))$.

Proposition 2 Let ζ be a nondecreasing function from $[0, \infty)$ to $[0, \infty]$ and let ψ be its pseudo-inverse. Let X be a metric space and $T : X \to X$. If $\zeta(d(Tx, Ty)) \leq d(x, y)$ $\forall x, y \in X$, then $d(Tx, Ty) \leq \psi(d(x, y))$ $\forall x, y \in X$; the converse is false.

Proof.
If $\zeta(d(Tx, Ty)) \leq d(x, y)$, then by the definition of pseudo-inverse $\psi(d(x, y)) \geq d(Tx, Ty)$.
To show that the converse is false, let

$$\zeta(s) = \begin{cases} 2s & \text{for } 0 \leq s \leq 1/2 \\ 1 & \text{for } 1/2 < s < 1 \\ \infty & \text{for } 1 \leq s < \infty \end{cases}$$

Then

$$\psi(t) = \begin{cases} t/2 & \text{for } 0 \le t < 1 \\ 1 & \text{for } 1 \le t < \infty \end{cases}$$

If X is any discrete metric space in which distance between any two distinct points is 1, and T the identity map, then T satisfies the inequality $d(Tx, Ty) \le \psi(d(x, y))$, but not the inequality $\zeta(d(Tx, Ty)) \le d(x, y)$.

Let (X, d) be a metric space and $T : X \to X$. The *modulus of uniform continuity* $\delta(\epsilon)$ of T is defined to be

$$\delta(\epsilon) = \sup\{\lambda : d(x, y) < \lambda \Rightarrow d(Tx, Ty) < \epsilon\}$$

for $\epsilon > 0$ and $\delta(0) = 0$.

Proposition 3

1. $\delta(\epsilon) = \inf\{d(x, y) : d(Tx, Ty) \ge \epsilon\}$.
2. $\delta(\epsilon)$ is nondecreasing and $0 \le \delta(\epsilon) \le \infty$.
3. $\delta(d(Tx, Ty)) \le d(x, y)$ $\forall x, y \in X$.

Proof.
Let $\mu(\epsilon) = \inf\{d(x, y) : d(Tx, Ty) \ge \epsilon\}$. Clearly $\mu(0) = 0 = \delta(0)$. Let $\epsilon > 0$. Call the set in the definition of $\delta(\epsilon)$ S. If $d(x_0, y_0) < \mu(\epsilon)$, then $d(x_0, y_0) \notin \{d(x, y) : d(Tx, Ty) \ge \epsilon\}$. So $d(Tx_0, Ty_0) < \epsilon$ and $\mu(\epsilon) \in S$. If $\lambda \in S$, then $d(Tx, Ty) \ge \epsilon \Rightarrow d(x, y) \ge \lambda$. Thus every number in the definition of $\mu(\epsilon)$ is $\ge \lambda$ and upon taking infimum $\mu(\epsilon) \ge \lambda$. This means $\mu(\epsilon) = \max S = \delta(\epsilon)$. That $\delta(\epsilon)$ is nondecreasing is obvious. Note that $\inf \emptyset = \infty$. Incidentally, we proved that for $\epsilon > 0$, the supremum in the definition of δ is actually the maximum.
If $d(Tx, Ty) = 0$, then (3) is obvious; so assume $d(Tx, Ty) = \epsilon > 0$. If $d(x, y) < \delta(\epsilon)$, then $d(Tx, Ty) < \epsilon$, a contradiction. So one must have $d(x, y) \ge \delta(\epsilon) = \delta(d(Tx, Ty))$.

Recall that T is *uniformly continuous* if

$$\forall \epsilon > 0 \ \exists \delta > 0 \text{ such that } d(x, y) < \delta \Rightarrow d(Tx, Ty) < \epsilon.$$

Proposition 4 Let X be a metric space and $T : X \to X$. Let $\delta(\epsilon)$ be the modulus of uniform continuity of T. The following are equivalent:

1. T is uniformly continuous.
2. $\delta(\epsilon) > 0$ for every $\epsilon > 0$.
3. There exists a nondecreasing function $\zeta : [0, \infty) \to [0, \infty]$ such that $\zeta(0) = 0, \zeta(\epsilon) > 0$ for every $\epsilon > 0$ and $\zeta(d(Tx, Ty)) \le d(x, y)$ for every $x, y \in X$.

4. There exists a function $\phi : [0,\infty) \to [0,\infty]$ such that $\phi(0) = 0, \phi$ is continuous at 0 and $d(Tx,Ty) \le \phi(d(x,y))\ \forall x,y \in X$.

In (3), one can choose ζ to be the modulus of uniform continuity of T. In (4), one can choose ϕ to be also nondecreasing and right continuous.

Proof.
(1) $\Rightarrow$ (2) follows from the definition. (2) $\Rightarrow$ (3): Take $\zeta = \delta$ and apply Proposition 3. (3) $\Rightarrow$ (4): Take ϕ to be the pseudo-inverse of ζ and apply Proposition 2. Since $\zeta(0) = 0$ and $\zeta(\epsilon) > 0$ for $\epsilon > 0$, one has $\phi(0) = 0$ by the definition of pseudo-inverse. ϕ is continuous at 0 by item 2 in Proposition 1.
(4) $\Rightarrow$ (1): $\forall\epsilon > 0, \exists\delta > 0$ such that $r < \delta \Rightarrow |\phi(r) - \phi(0)| = \phi(r) < \epsilon$. Therefore $d(x,y) < \delta \Rightarrow d(Tx,Ty) \le \phi(d(x,y)) < \epsilon$.

Definition 2 A function $\lambda : [0,\infty) \to [0,\infty)$ will be called an *L-function* if $\lambda(0) = 0, \lambda(s) > 0\ \forall s > 0$, and for every $s > 0$, there exists $u > s$ such that

$$\lambda(t) \le s \text{ for } t \in [s,u]$$

Note that every L-function satisfies $\lambda(s) \le s\ \forall s \ge 0$.

Proposition 5 Let $\zeta : [0,\infty) \to [0,\infty]$ be nondecreasing,$\zeta(0) = 0, \zeta(s) > s$ for $s > 0$. Let ϕ be its pseudo-inverse. Then

1. $\phi(t) \le t\ \forall t \in [0,\infty)$.
2. ϕ is an L-function except that $\phi(s)$ could be 0 for some $s > 0$.

Proof.
(1): We have $\phi(0) = 0$. If $t > 0$ and $\zeta(s) \le t$, then $s < t$ (for if $s \ge t$, then $\zeta(s) \ge \zeta(t) > t$). So $\phi(t) = \sup\{s : \zeta(s) \le t\} \le t$.
(2): Let $t > 0$. Suppose $\phi(t) < t$. Then by the right continuity of ϕ there exists $u > t$ such that $\phi(s) < t\ \forall s \in [t,u]$.
If $\phi(t) = t$, then $\sup\{s : \zeta(s) \le t\} = t$. This implies that $\zeta(t-) = t$. But $\zeta(t) > t$ by assumption. So ζ has a discontinuity at t. By item 3 in Proposition 1, there exists $u > t$ such that $\phi(s) = t\ \forall s \in [t,u]$.
If ζ is discontinuous at 0, then $\phi(s) = 0\ \forall s \in [0, \zeta(0+))$.

The proof of (3) $\Rightarrow$ (1) in the following theorem is due to Wong [4]; we present it here for completeness.

Theorem 1 *Let X be a metric space. Let $T : X \to X$ and let $\delta(\epsilon)$ be its modulus of uniform continuity. The following are equivalent:*

1. *T satisfies the Meir-Keeler's condition (1).*
2. *$\delta(\epsilon) > \epsilon\ \forall\epsilon > 0$.*

3. [4] *There exists a right lower semicontinuous function* $\zeta : [0,\infty) \to [0,\infty)$ *such that* $\zeta(0) = 0, \zeta(\epsilon) > \epsilon$ *for every* $\epsilon > 0$ *and* $\zeta(d(Tx,Ty)) \leq d(x,y)$ *for every* $x, y \in X$.

4. *There exists an L-function* $\phi : [0,\infty) \to [0,\infty)$ *such that* $d(Tx,Ty) < \phi(d(x,y))\ \forall x \neq y \in X$.

In (4) one can choose ϕ *to be also nondecreasing and right continuous.*

Proof.
(1) $\Rightarrow$ (2): (1) implies that $d(Tx,Ty) < d(x,y)\ \forall x \neq y \in X$. Thus Meir-Keeler's condition (1) is valid even if $d(x,y) < \epsilon$. Then by definition of $\delta(\epsilon)$ one has $\delta(\epsilon) > \epsilon$.
(2) $\Rightarrow$ (3): If $\delta(\epsilon) < \infty$, one can take $\zeta = \delta$ and apply Proposition 3. If $\delta(\epsilon) = \infty$ for some $\epsilon > 0$, let $\epsilon_0 = \inf\{\epsilon : \delta(\epsilon) = \infty\}$.
Case (i): $\delta(\epsilon_0) < \infty$. Define $\zeta(\epsilon) = \delta(\epsilon)$ for $\epsilon \leq \epsilon_0$ and $\zeta(\epsilon) = \delta(\epsilon_0) + 2(\epsilon - \epsilon_0)$ for $\epsilon > \epsilon_0$.
Case (ii): $\delta(\epsilon_0) = \infty$. Define $\zeta(\epsilon) = \delta(\epsilon)$ for $\epsilon < \epsilon_0, \zeta(\epsilon_0) = 2\epsilon_0$, and $\zeta(\epsilon) = 2\epsilon_0 + 2(\epsilon - \epsilon_0)$ for $\epsilon > \epsilon_0$.
Since $\delta(\epsilon)$ is nondecreasing and hence right lower semicontinuous, it is easy to see that in each case ζ is right lower semicontinuous. By Proposition 3, $\delta(d(Tx,Ty)) \leq d(x,y)\ \forall x,y \in X$. Since the left-hand side of this inequality cannot be infinity, one has $d(Tx,Ty) \leq \epsilon_0\ \forall x,y \in X$ in Case (i) and $d(Tx,Ty) < \epsilon_0\ \forall x,y \in X$ in Case (ii). Therefore $\zeta(d(Tx,Ty)) = \delta(d(Tx,Ty)) \leq d(x,y)\ \forall x,y \in X$.
(3) $\Rightarrow$ (1): ([4]) Let $\epsilon > 0$. By the right lower-semicontinuity of ζ, there exists $\lambda_1 > 0$ such that

$$\frac{\epsilon + \zeta(\epsilon)}{2} < \zeta(s)\ \forall \epsilon \leq s < \epsilon + \lambda_1. \tag{2}$$

Let $\lambda = \min\{\lambda_1, (\zeta(\epsilon) - \epsilon)/2\}$. (3) implies that $\lambda > 0$ and that $d(Tx,Ty) < d(x,y)\ \forall x \neq y \in X$. Suppose $\epsilon \leq d(x,y) < \epsilon + \lambda$. If $d(Tx,Ty) \geq \epsilon$, then $\epsilon \leq d(Tx,Ty) < d(x,y) < \epsilon + \lambda_1$ and by (2)

$$\begin{aligned}
\frac{\epsilon + \zeta(\epsilon)}{2} &< \zeta(d(Tx,Ty)) \\
&\leq d(x,y) \\
&< \epsilon + \frac{\zeta(\epsilon) - \epsilon}{2} \\
&= \frac{\epsilon + \zeta(\epsilon)}{2},
\end{aligned}$$

a contradiction. Hence $d(Tx,Ty) < \epsilon$.
(2) $\Rightarrow$ (4): Let $\beta = \inf\{\epsilon : \delta(\epsilon) = \infty\}$ ($\inf \emptyset = \infty$). Let $\alpha(\epsilon) = (\epsilon + \delta(\epsilon))/2$ and let ϕ_0 and ψ be the pseudo-inverses of α and δ respectively. Obviously, $\alpha(\beta+) = \delta(\beta+) = \infty$.
If $\beta = 0$, then T is a constant map and we may take ϕ to be the zero function. So assume that $\beta > 0$.

First we consider $\epsilon < \beta$. Clearly $\alpha(\epsilon) < \delta(\epsilon)\ \forall \beta > \epsilon > 0$. If δ is continuous at 0, then for every $\epsilon > 0$, there exists $d > 0$ such that $\delta(s) < \epsilon\ \forall s < d$; so by the definition of ψ, one has $\psi(\epsilon) \geq d > 0$ and hence $\phi_0(\epsilon) \geq \psi(\epsilon) > 0$. If δ is discontinuous at 0, then so is α and by item 3 in Proposition 1, $\phi_0(t) = 0 = \psi(t)\ \forall t \in [0, \alpha(0+))$.

Suppose $\epsilon > 0$ and δ is continuous at ϵ. Let $t = \alpha(\epsilon)$. For every $\epsilon_1 > \epsilon$, one has

$$\alpha(\epsilon_1) = \frac{\epsilon_1 + \delta(\epsilon_1)}{2} > \frac{\epsilon + \delta(\epsilon)}{2} = \alpha(\epsilon);$$

so $\phi_0(t) = \epsilon$ by the definition of pseudo-inverse. Since $\delta(\epsilon) > \alpha(\epsilon) = t$, by the continuity of δ at ϵ, there exists $\epsilon_1 < \epsilon$ such that $\delta(\epsilon_1) > t$. Hence $\psi(t) \leq \epsilon_1 < \epsilon = \phi_0(t)$.

Suppose $\epsilon > 0$ and δ is discontinuous at ϵ. Then α is also discontinuous at ϵ. Case (i): $\alpha(\epsilon+) \leq \delta(\epsilon-)$. Let $t \in [\alpha(\epsilon-), \alpha(\epsilon+))$. Then $t < \delta(\epsilon-)$ and there exists $\epsilon_0 < \epsilon$ such that $\delta(s) > t\ \forall s > \epsilon_0$. Therefore by the definition of ψ, $\psi(t) < \epsilon$. By the item 3 in Proposition 1, $\phi_0(t) = \epsilon\ \forall t \in [\alpha(\epsilon-), \alpha(\epsilon+))$. Thus $\phi_0(t) = \epsilon > \psi(t)\ \forall t \in [\alpha(\epsilon-), \alpha(\epsilon+))$. Case (ii): $\alpha(\epsilon+) > \delta(\epsilon-)$. Then on the interval $[\alpha(\epsilon-), \delta(\epsilon-))$, $\phi_0(t) = \epsilon > \psi(t)$ as in Case (i). Also note that $\alpha(\epsilon+) \leq \delta(\epsilon+)$. Then by applying the item 3 in Proposition 1 to both α and δ, we get $\psi(t) = \epsilon = \phi_0(t)\ \forall t \in [\delta(\epsilon-), \alpha(\epsilon+))$.

The above paragraph is also valid if $\beta < \infty$ and $\epsilon = \beta$.

To summarize, $\phi_0(t) \geq \psi(t)$ and the equality holds only when t is 0 or in an interval of the form $[\delta(\epsilon-), \alpha(\epsilon+))$ for some $\epsilon \geq 0$ where δ is discontinuous ($\delta(0-)$ is taken to be 0). Moreover, in the case of equality, $\psi(t) = \phi_0(t) = \epsilon\ \forall t \in [\delta(\epsilon-), \alpha(\epsilon+))$.

Now we need to modify ϕ_0 slightly to obtain ϕ.

If δ is discontinuous at 0, we define ϕ to be linear on the interval $[0, \alpha(0+))$ with $\phi(0) = 0$ and $\phi(\alpha(0+)-) = \frac{1}{2}\alpha(0+)$. If δ is discontinuous at $\epsilon > 0$ and $\delta(\epsilon-) > \epsilon$ we define $\phi(t) = \frac{\epsilon+\delta(\epsilon-)}{2}$ for $t \in [\delta(\epsilon-), \alpha(\epsilon+))$. Suppose δ is discontinuous at $\epsilon > 0$ and $\delta(\epsilon-) = \epsilon$. We consider the following two cases. Case (i): $\alpha(\epsilon+) \leq \delta(\epsilon)$; define $\phi(t) = \phi_0(t)$ on the interval $[\epsilon, \alpha(\epsilon+))$. Case (ii): $\alpha(\epsilon+) > \delta(\epsilon)$; define $\phi(t) = \phi_0(t)$ on the interval $[\epsilon, \delta(\epsilon))$ and $\phi(t) = \frac{\epsilon+\delta(\epsilon)}{2}\ \forall t \in [\delta(\epsilon), \alpha(\epsilon+))$. Define $\phi(t) = \phi_0(t)$ for any other s. The function ϕ so defined satisfies $\phi(t) > 0\ \forall t > 0$, $\phi(t) \leq t, \phi(t) \geq \psi(t)\ \forall t \geq 0$ and the equality holds only when $t = 0$ or when $\delta(\epsilon-) = \epsilon, \phi(t) = \psi(t) = \epsilon\ \forall t \in [\epsilon, \min\{\delta(\epsilon), \alpha(\epsilon+)\})$.

Since ϕ_0 is right continuous by Proposition 1, it is easy to see that ϕ is also right continuous. It follows from the proof of Proposition 5 that ϕ is an L-function.

By Proposition 2, $d(Tx, Ty) \leq \psi(d(x, y)) < \phi(d(x, y))\ \forall x, y \in X$ except when $d(x, y) = 0$ or $d(x, y) \in [\epsilon, \min\{\delta(\epsilon), \alpha(\epsilon+)\})$ for some $\epsilon > 0$ such that $\delta(\epsilon-) = \epsilon$. In the latter case, since $d(x, y) < \delta(\epsilon)$, we have $d(Tx, Ty) < \epsilon = \phi(d(x, y))$. This completes the proof that (2) implies (4).

The function ϕ above may not be nondecreasing. If we define $\xi(s) = \sup\{\phi(t) : t \leq s\}$, then ξ is a nondecreasing, right continuous L-function that can be used in place of ϕ.

(4) $\Rightarrow$ (1): Suppose $d(Tx, Ty) < \phi(d(x, y))$ for $x \neq y \in X$, for some L-function

ϕ. For every $\epsilon > 0$, there exists $\delta > 0$ such that $\phi(t) \leq \epsilon$ $\forall t \in [\epsilon, \epsilon + \delta)$. So if $\epsilon \leq d(x,y) < \epsilon + \delta$, then $d(Tx,Ty) < \phi(d(x,y)) \leq \epsilon$.

Remark 1 It is easy to see that the function ϕ in the Boyd-Wong's theorem is an L-function. However, the inequality there is not strict. To get strict inequality as in item 4, simply replace the function $\phi(s)$ there by $\frac{s+\phi(s)}{2}$.

Remark 2 It is not necessary to require that $\phi(s) > 0$ $\forall s > 0$ in Theorem 1 if one drops the condition from the definintion of L-function and change the condition

$$d(Tx,Ty) < \phi(d(x,y)) \ \forall x \neq y \in X$$

to

$$d(Tx,Ty) < \phi(d(x,y)) \ \forall x,y \in X \text{ with } \phi(d(x,y)) > 0$$

Remark 3 Let $X = [0,1] \cup \{3n, 3n+1\}_{n=1}^{\infty}$ with the Euclidean metric and $T : X \to X$ the map defined by ([1])

$$T(x) = \begin{cases} x/2 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } x = 3n \\ 1 - \frac{1}{n+2} & \text{for } x = 3n+1 \end{cases}$$

Then

$$\delta(\epsilon) = \begin{cases} 2\epsilon & \text{for } 0 \leq \epsilon \leq 1/2 \\ 1 & \text{for } 1/2 < \epsilon < 1 \\ \infty & \text{for } 1 \leq \epsilon < \infty \end{cases}$$

$$\psi(t) = \begin{cases} t/2 & \text{for } 0 \leq t < 1 \\ 1 & \text{for } 1 \leq t < \infty \end{cases}$$

and

$$\phi(t) = \phi_0(t) = \begin{cases} \frac{2}{3}t & \text{for } 0 \leq t \leq 3/4 \\ 2t-1 & \text{for } 3/4 < t \leq 1 \\ 1 & \text{for } 1 \leq t < \infty \end{cases}$$

Note that $\phi(1) = 1$. Indeed, for any L-function ϕ satisfying $d(Tx,Ty) < \phi(d(x,y))$ $\forall x \neq y \in X$, we have, by setting $x = 3n, y = 3n+1$, $1 - \frac{1}{n+2} < \phi(1)$. Thus $1 \leq \phi(1)$. But $1 \geq \phi(1)$ by definition. Hence $\phi(1) = 1$. This shows that it is not always possible to find a ϕ such that ϕ^n converges pointwise to 0.

Let (X,d) be a metric space and $T : X \to 2^X \backslash \{\emptyset\}$ a multivalued map with Tx closed for every $x \in X$. Let H be the Hausdorff metric on nonempty subsets of H, i.e.

$$H(A,B) = \max\{\sup_{a \in A} d(a,B), \sup_{b \in B} d(b,A)\}$$

where $d(x, A) = \inf_{a \in A} d(x, a)$. Meir-Keeler's condition and modulus of uniform continuity of T can be defined similarly:

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ such that } \epsilon \leq d(x, y) < \epsilon + \delta \Rightarrow H(Tx, Ty) < \epsilon \tag{3}$$

$$\delta(\epsilon) = \inf\{d(x, y) : H(Tx, Ty) \geq \epsilon\} \tag{4}$$

The following theorem remains valid:

Theorem 2 *Let X be a metric space. Let $T : X \to 2^X \setminus \{\emptyset\}$ and let $\delta(\epsilon)$ be its modulus of uniform continuity. The following are equivalent:*

1. *T satisfies condition (3).*
2. *$\delta(\epsilon) > \epsilon \; \forall \epsilon > 0$.*
3. *There exists a right lower semicontinuous function $\zeta : [0, \infty) \to [0, \infty)$ such that $\zeta(0) = 0, \zeta(\epsilon) > \epsilon$ for every $\epsilon > 0$ and $\zeta(H(Tx, Ty)) \leq d(x, y)$ for every $x, y \in X$.*
4. *There exists an L-function $\phi : [0, \infty) \to [0, \infty)$ such that $H(Tx, Ty) < \phi(d(x, y)) \; \forall x \neq y \in X$.*

In (4) one can choose ϕ to be also nondecreasing and right continuous.

Open Problem

1. Let (X, d) be a complete metric space and $T : X \to 2^X \setminus \{\emptyset\}$ a multivalued mapping such that Tx is closed for every x and

$$H(Tx, Ty) < \psi(d(x, y)), \forall \; x \neq y \in X$$

where H denotes the Hausdorff metric and ψ is an L-function. Does T have a fixed point?
The answer is yes if Tx is compact for every x (Reich [3]).

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