



## FIXED POINT THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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### 1. INTRODUCTION

LET  $X$  be a Banach space and  $C$  be a nonempty subset of  $X$ . Then a mapping  $T: C \rightarrow C$  is said to be a Lipschitzian mapping if, for each integer  $n \geq 1$ , there exists a constant  $k_n > 0$  such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all  $x, y$  in  $C$ . A Lipschitzian mapping  $T$  is called uniformly  $k$ -Lipschitzian if  $k_n = k$  for all  $n \geq 1$ , nonexpansive if  $k_n = 1$  for all  $n \geq 1$ , and asymptotically nonexpansive if  $\lim_n k_n = 1$ , respectively.

In 1965, Kirk [1] proved that if  $C$  is a weakly compact convex subset of a Banach space with normal structure, then every nonexpansive self-mapping  $T$  of  $C$  has a fixed point. (A nonempty convex subset  $C$  of a normed linear space is said to have normal structure if each bounded convex subset  $K$  of  $C$  consisting of more than one point contains a nondiametral point, i.e. an  $x \in K$  such that  $\sup\{\|x - y\|: y \in K\} < \sup\{\|u - v\|: u, v \in K\} = \text{diam } K$ .) Seven years later, in 1972, Goebel and Kirk [2] proved that if the space  $X$  is further assumed to be uniformly convex, then every asymptotically nonexpansive self-mapping  $T$  of  $C$  has a fixed point. This result has now been generalized to a 2-uniformly rotund Banach space by Yu and Dai [3] and generally to a  $k$ -uniformly rotund ( $k$ -UR) Banach space for any integer  $k \geq 1$  independently by Martinez Yanez [4] and Xu [5], and more generally to a nearly uniformly convex (NUC) Banach space by the latter author [6]. (We refer the reader to [7, 8] for definitions of  $k$ -UR and NUC Banach spaces.) However, it remains an open question whether normal structure implies the existence of fixed points of asymptotically nonexpansive mappings. In this paper we present some results on fixed points of asymptotically nonexpansive mappings. Precisely, we prove in Section 2 the strong convergence (under certain assumptions) of an approximating fixed point for an asymptotically nonexpansive mapping in a uniformly smooth Banach space. This partially extends a celebrated convergence theorem due to Reich [9] for nonexpansive mappings. In Section 3 we verify the existence and weak convergence of fixed points of asymptotically nonexpansive mappings in a space with a weakly continuous duality map. Finally, in Section 4, we provide two fixed point theorems for asymptotically nonexpansive mappings which connect with Maluta's constant for a Banach space.

## 2. UNIFORMLY SMOOTH BANACH SPACES

Let  $X$  be a Banach space. Recall that  $X$  is said to be smooth if, for each  $x \in S_X$ , the unit sphere of  $X$ , the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.1)$$

exists for all  $y \in S_X$ . If the limit (2.1) exists and is attained uniformly in  $x, y \in S_X$ , then  $X$  is said to be uniformly smooth. In this case, the (normalized) duality map  $J: X \rightarrow X^*$ , the dual space of  $X$ , defined by

$$J(x) = \{x^* \in X^*: \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X$$

is single-valued and uniformly continuous on bounded subsets of  $X$  when both  $X$  and  $X^*$  are endowed with their norm topologies.

Now suppose  $X$  is a uniformly smooth Banach space,  $C$  is a bounded closed convex subset of  $X$ , and  $T: C \rightarrow C$  is a nonexpansive mapping. Then to a fixed  $u \in C$  and each integer  $n \geq 1$ , by the Banach Contraction Principle, we have a unique  $x_n \in C$  such that

$$x_n = \frac{1}{n}u + \left(1 - \frac{1}{n}\right)Tx_n.$$

(Such a sequence  $\{x_n\}$  is said to be an approximating fixed point of  $T$  since it possesses the property that  $\lim_n \|Tx_n - x_n\| = 0$ .) The celebrated convergence theorem of Reich [9] now states that  $\{x_n\}$  does converge strongly to a fixed point of  $T$ . In this section we shall partially extend this result to asymptotically nonexpansive mappings. But first we begin with some preliminaries.

Let  $E$  be a nonempty bounded closed convex subset of a Banach space  $X$  and let  $d(E) = \sup\{\|x - y\|: x, y \in E\}$  be the diameter of  $E$ . For each  $x \in E$ , let  $r(x, E) = \sup\{\|x - y\|: y \in E\}$  and let  $r(E) = \inf\{r(x, E): x \in E\}$ , the Chebyshev radius of  $E$  relative to itself. The normal structure coefficient of  $X$  is defined [10] as the number

$$N(X) = \inf\{d(E)/r(E): E \text{ bounded closed convex subset of } X \text{ with } d(E) > 0\}.$$

A space  $X$  such that  $N(X) > 1$  is said to have uniformly normal structure. It is known that a space with uniformly normal structure is reflexive and that all uniformly convex or uniformly smooth Banach spaces have uniformly normal structure (cf. for example [11]). Now we prove the following result that is a slight generalization of a theorem of Casini and Maluta [12].

**THEOREM 1.** Suppose  $X$  is a Banach space with uniformly normal structure,  $C$  is a nonempty bounded subset of  $X$ , and  $T: C \rightarrow C$  is a uniformly  $k$ -Lipschitzian mapping with  $k < N(X)^{1/2}$ . Suppose also there exists a nonempty bounded closed convex subset  $E$  of  $C$  with the following property (P):

$$(P) \quad x \in E \quad \text{implies } \omega_w(x) \subset E,$$

where  $\omega_w(x)$  is the weak  $\omega$ -limit set of  $T$  at  $x$ , i.e. the set

$$\{y \in X: y = \text{weak-lim}_j T^{n_j}x \text{ for some } n_j \uparrow \infty\}.$$

Then  $T$  has a fixed point in  $E$ .

*Proof.* Take any  $x_0$  in  $E$  and consider, for each integer  $n \geq 1$ , the sequence  $\{T^j x_0\}_{j \geq n}$ . According to the definition of  $N(X)$ , we have a  $y_n \in \overline{\text{co}}\{T^j x_0 : j \geq n\}$  ( $\overline{\text{co}}$  denotes the closed convex hull) such that

$$\overline{\lim}_j \|T^j x_0 - y_n\| \leq \tilde{N}(X)A(\{T^j x_0\}_{j \geq n}), \quad (2.2)$$

where  $\tilde{N}(X) = N(X)^{-1}$  and  $A(\{z_n\})$  denotes the asymptotic diameter of the sequence  $\{z_n\}$ , i.e. the number

$$\lim_n (\sup\{\|z_i - z_j\| : i, j \geq n\}).$$

Since  $X$  is reflexive,  $\{y_n\}$  admits a subsequence  $\{y_{n_i}\}$  converging weakly to some  $x_1 \in X$ . From (2.2) and the  $w$ -l.s.c. of the functional  $\overline{\lim}_n \|T^n x_0 - y\|$ , it follows that

$$\overline{\lim}_n \|T^n x_0 - x_1\| \leq \tilde{N}(X)A(\{T^n x_0\}). \quad (2.3)$$

It is also easily seen that  $x_1$  belongs to the set  $\bigcap_{n=1}^{\infty} \overline{\text{co}}\{T^j x_0 : j \geq n\}$  and that

$$\|z - x_1\| \leq \overline{\lim}_n \|z - T^n x_0\| \quad \text{for all } z \in X. \quad (2.4)$$

Observing property (P) and the fact that  $\bigcap_{n=1}^{\infty} \overline{\text{co}}\{T^j x_0 : j \geq n\} = \overline{\text{co}} \omega_w(x)$  which is easy to prove by using the Separation Theorem (cf. [13]), we get that  $x_1$  actually lies in  $E$ . So we can repeat the above process and obtain a sequence  $\{x_n\}_{n=1}^{\infty}$  in  $E$  with the properties: for all integers  $m \geq 1$ ,

$$\overline{\lim}_n \|T^n x_{m-1} - x_m\| \leq \tilde{N}(X)A(\{T^n x_{m-1}\}), \quad (2.5)$$

$$\|z - x_m\| \leq \overline{\lim}_n \|z - T^n x_{m-1}\| \quad \text{for all } z \in X. \quad (2.6)$$

Now proceeding with the same argument as in the proof of Casini and Maluta [12], we see that  $\{x_n\}$  converges strongly to a fixed point of  $T$ . ■

A direct consequence of theorem 1 is the following result that will be used in the proof of theorem 2 below.

**COROLLARY 1.** Suppose  $X$  is a uniformly smooth Banach space,  $C$  is a nonempty bounded subset of  $X$ , and  $T: C \rightarrow C$  is an asymptotically nonexpansive mapping. Suppose also there exists a nonempty closed bounded convex subset  $E$  of  $C$  with the property (P) above. Then  $T$  has a fixed point in  $E$ .

Suppose now  $C$  is a bounded closed convex subset of a Banach space  $X$  and  $T: C \rightarrow C$  is an asymptotically nonexpansive mapping (we may always assume  $k_n \geq 1$  for all  $n \geq 1$ ). Fix a  $u$  in  $C$  and define for each integer  $n \geq 1$  the contraction  $S_n: C \rightarrow C$  by

$$S_n(x) = \left(1 - \frac{t_n}{k_n}\right)u + \frac{t_n}{k_n} T^n x,$$

where  $\{t_n\} \subset [0, 1)$  is any sequence such that  $t_n \rightarrow 1$ . Then the Banach Contraction Principle yields a unique point  $x_n$  fixed by  $S_n$ . Now the question gives rise to whether the sequence  $\{x_n\}$  converges strongly to a fixed point of  $T$ . The following is a partial answer.

THEOREM 2. Suppose  $X$  is uniformly smooth and  $\{t_n\}$  is chosen so that

$$\lim_n (k_n - 1)/(k_n - t_n) = 0.$$

(Such a sequence  $\{t_n\}$  always exists, for example, take  $t_n = \min\{1 - (k_n - 1)^{1/2}, 1 - n^{-1}\}$ .) Suppose in addition the following condition

$$\lim_n \|x_n - Tx_n\| = 0 \quad (2.7)$$

holds. Then the sequence  $\{x_n\}$  converges strongly to a fixed point of  $T$ .

*Proof.* From corollary 1, the fixed point set  $F(T)$  of  $T$  is nonempty. For each  $v \in F(T)$ , we have

$$\begin{aligned} \langle x_n - T^n x_n, J(x_n - v) \rangle &= \langle x_n - v, J(x_n - v) \rangle + \langle v - T^n x_n, J(x_n - v) \rangle \\ &\geq \|x_n - v\|^2 - \|v - T^n x_n\| \|x_n - v\| \\ &\geq -(k_n - 1) \|x_n - v\|^2 \\ &\geq -(k_n - 1) d^2, \end{aligned}$$

where  $d = \text{diam } C$ . Since  $x_n$  is a fixed point of  $S_n$ , it follows that

$$x_n - T^n x_n = \frac{k_n - t_n}{t_n} (u - x_n)$$

and thus from the last inequality above, we get

$$\langle x_n - u, J(x_n - v) \rangle \leq s_n d^2, \quad (2.8)$$

where  $s_n = t_n(k_n - 1)/(k_n - t_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Now let LIM be a Banach limit and define  $f: C \rightarrow [0, \infty)$  by

$$f(z) = \text{LIM}_n \|x_n - z\|^2.$$

Since  $f$  is continuous and convex,  $f(z) \rightarrow \infty$  as  $\|z\| \rightarrow \infty$ , and  $X$  is reflexive,  $f$  attains its infimum over  $C$ . Hence, the set

$$E = \left\{ x \in C: f(x) = \min_{z \in C} f(z) \right\}$$

is nonempty, closed and convex. Though  $E$  is not necessarily invariant under  $T$ , it does have the property (P). In fact, if  $x$  is in  $E$  and  $y = w\text{-}\lim_j T^{m_j} x$  belongs to the weak  $\omega$ -limit set  $\omega_w(x)$  of  $T$  at  $x$ , then from the  $w$ -l.s.c. of  $f$  and (2.7), we have

$$\begin{aligned} f(y) &\leq \underline{\lim}_j f(T^{m_j} x) \leq \overline{\lim}_m f(T^m x) \\ &= \overline{\lim}_m (\text{LIM}_n \|x_n - T^m x_n\|^2) = \overline{\lim}_m (\text{LIM}_n \|T^m x_n - T^m x\|^2) \\ &\leq \overline{\lim}_m k_m^2 \text{LIM}_n \|x_n - x\|^2 = \text{LIM}_n \|x_n - x\|^2 \\ &= \min_{z \in C} f(z). \end{aligned}$$

This shows that  $y$  belongs to  $E$  and hence  $E$  satisfies the property (P). It follows from corollary 1 that  $T$  has a fixed  $z \in E$ . Since  $z$  is also a minimizer of  $f$  over  $C$ , it follows that

$$\lim_{t \rightarrow 0^+} \frac{f(z + t(x - z)) - f(z)}{t} \geq 0$$

for all  $x \in C$ . This, together with the uniform smoothness of  $X$ , easily implies that (see [14, pp. 46-47] for details)

$$\text{LIM}_n \langle x - z, J(x_n - z) \rangle \leq 0 \quad (2.9)$$

for all  $x \in C$ ; in particular, we have

$$\text{LIM}_n \langle u - z, J(x_n - z) \rangle \leq 0. \quad (2.10)$$

Combining (2.10) and (1.8), we get

$$\text{LIM}_n \langle x_n - z, J(x_n - z) \rangle = \text{LIM}_n \|x_n - z\|^2 \leq 0.$$

Therefore, there is a subsequence  $\{x_{n_j}\}$  of  $\{x_n\}$  which converges strongly to  $v$ . To complete the proof, suppose there is another subsequence  $\{x_{m_j}\}$  of  $\{x_n\}$  which converges strongly to (say)  $y$ . Then  $y$  is a fixed point of  $T$  by hypothesis (2.7). It then follows from (2.8) that

$$\langle z - u, J(z - y) \rangle \leq 0,$$

and

$$\langle y - u, J(y - z) \rangle \leq 0.$$

Adding these two inequalities yields

$$\langle z - y, J(z - y) \rangle = \|z - y\|^2 = 0$$

and thus  $z = y$ . This proves the strong convergence of  $\{x_n\}$  to  $z$ . ■

### 3. WEAKLY CONTINUOUS DUALITY MAP

Recall that a Banach space  $X$  is said to satisfy Opial's condition [15] if, for any sequence  $\{x_n\}$  in  $X$ , the condition that  $\{x_n\}$  converges weakly to  $x \in X$  implies that

$$\overline{\lim}_n \|x_n - x\| < \overline{\lim}_n \|x_n - y\|$$

for all  $y \in X$ ,  $y \neq x$ . It has been proved that Opial's condition implies weakly normal structure and, hence, the fixed point property for nonexpansive mappings. However, it is not clear whether Opial's condition implies the fixed point property for asymptotically nonexpansive mappings. Theorem 3 of this section will provide a partial answer to this question.

By a gauge we mean a continuous strictly increasing function  $\varphi$  defined  $R^+ := [0, \infty)$  such that  $\varphi(0) = 0$  and  $\lim_{r \rightarrow \infty} \varphi(r) = \infty$ . We associate with a gauge  $\varphi$  a (generally multivalued) duality map  $J_\varphi : X \rightarrow X^*$  defined by

$$J_\varphi(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|\varphi(\|x\|) \text{ and } \|x^*\| = \varphi(\|x\|)\}.$$

Clearly the (normalized) duality map  $J$  corresponds to the gauge  $\varphi(t) = t$ . Browder [16] initiated the study of certain classes of nonlinear operators by means of a duality map  $J_\varphi$ . Set for  $t \geq 0$ ,

$$\Phi(t) = \int_0^t \varphi(r) \, dr.$$

Then it is known that  $J_\varphi(x)$  is the subdifferential of the convex function  $\Phi(\|\cdot\|)$  at  $x$ . Now recall that  $X$  is said to have a weakly continuous duality map if there exists a gauge  $\varphi$  such that the duality map  $J_\varphi$  is single-valued and continuous from  $X$  with the weak topology to  $X^*$  with the weak\* topology. A space with a weakly continuous duality map is easily seen to satisfy Opial's condition (cf. [16]). Every  $l^p$  ( $1 < p < \infty$ ) space has a weakly continuous duality map with the gauge  $\varphi(t) = t^{p-1}$ .

**THEOREM 3.** Suppose  $X$  is a Banach space with a weakly continuous duality map  $J_\varphi$ ,  $C$  is a weakly compact convex subset of  $X$ , and  $T: C \rightarrow C$  is an asymptotically nonexpansive mapping. Then we have the following conclusions:

- (i)  $T$  has a fixed point in  $C$ ; and
- (ii) if  $T$  is weakly asymptotically regular at some  $x \in C$ , that is,  $w\text{-}\lim_n (T^n x - T^{n+1} x) = 0$ , then  $\{T^n x\}$  converges weakly to a fixed point of  $T$ .

*Proof.* (i) Let  $F$  be the family of subsets  $K$  of  $C$  which are nonempty, closed, convex, and satisfy the following property

$$(P) \quad x \in K \quad \text{implies} \quad \omega_w(x) \subset K.$$

(Here, as before,  $\omega_w(x)$  is the weak  $\omega$ -limit set of  $T$  at  $x$ .)  $F$  is then ordered by inclusion. The weak compactness of  $C$  now allows one to use Zorn's lemma to obtain a minimal element (say)  $K$  in  $F$ . For each  $x$  in  $C$ , define the functional  $r_x$  by

$$r_x(y) = \overline{\lim}_n \|T^n x - y\|.$$

Then by lemma 1 of [6], when  $x$  lies in  $K$ ,  $r_x$  is a constant over  $y \in K$  and this constant is independent of  $x \in K$ ; that is,

$$\overline{\lim}_n \|T^n x - y\| = r \quad \text{for all } x, y \in K. \tag{3.1}$$

Now fix  $x$  in  $K$  and let  $\{T^{n_i} x\}$  be a subsequence of  $\{T^n x\}$  converging weakly to some  $y$  that is in  $K$  by property (P) and such that  $r' := \lim_i \|T^{n_i} x - y\|$  exists. For any integers  $n, m \geq 1$ , noting the identity

$$\Phi(\|x + y\|) = \Phi(\|x\|) + \int_0^1 \langle y, J_\varphi(x + ty) \rangle \, dt$$

for all  $x, y \in X$ , we have

$$\begin{aligned} \Phi(\|T^n x - T^m x\|) &= \Phi(\|(T^n x - y) + (y - T^m x)\|) \\ &= \Phi(\|T^n x - y\|) + \int_0^1 \langle y - T^m x, J_\varphi(T^n x - y + t(y - T^m x)) \rangle \, dt. \end{aligned}$$

Substituting  $n_i$  for  $n$  and letting  $i$  go to infinity, we get

$$\begin{aligned}\lim_i \Phi(\|T^{n_i}x - T^m x\|) &= \Phi(r') + \int_0^1 \langle y - T^m x, J_\varphi(t(y - T^m x)) \rangle dt \\ &= \Phi(r') + \int_0^1 \|y - T^m x\| \varphi(t\|y - T^m x\|) dt \\ &= \Phi(r') + \Phi(\|y - T^m x\|).\end{aligned}$$

It follows that

$$\begin{aligned}\Phi(r') + \Phi(r) &= \overline{\lim}_m \left( \lim_i \Phi(\|T^{n_i}x - T^m x\|) \right) \\ &\leq \overline{\lim}_m \left( \overline{\lim}_n \Phi(\|T^n x - T^m x\|) \right) \\ &\leq \overline{\lim}_m \left( \overline{\lim}_n \Phi(k_m \|T^{n-m}x - x\|) \right) \\ &= \overline{\lim}_n \Phi(\|T^n x - x\|) = \Phi(r),\end{aligned}$$

which implies that  $\Phi(r') = 0$ . Hence,  $r' = 0$ , i.e.  $\{T^{n_i}x\}$  strongly converges to  $y$ . This proves that, for each  $x$  in  $K$ , the strong  $\omega$ -limit set  $\omega(x) := \{y \in X: y\text{-strong-}\lim_i T^{n_i}x \text{ for some } n_i \uparrow \infty\}$  of  $T$  at  $x$  is nonempty. It is clearly closed. We further claim that  $\omega(x)$  is norm-compact. In fact, given any sequence  $\{u_j\}$  in  $\omega(x)$ . It is easy to construct a subsequence  $\{T^{m_j}x\}$  of  $\{T^n x\}$  such that  $\|T^{m_j}x - u_j\| < j^{-1}$  for all  $j \geq 1$ . Repeating the argument above, we get a subsequence  $\{T^{m'_j}x\}$  of  $\{T^{m_j}x\}$  converging strongly to some  $z \in \omega(x)$ . Hence,  $u_j \rightarrow z$  strongly indicating the norm-compactness of  $\omega(x)$ . Now by lemma 2 of [6],  $T$  has a fixed point and (i) is thus proven.

Now we turn to the proof of (ii). First observe that for any  $p \in F(T)$ , the  $\lim_n \|T^n x - p\|$  exists. In fact, for all integers  $n, m \geq 1$ , we have

$$\|T^{n+m}x - p\| = \|T^n(T^m x) - T^n p\| \leq k_n \|T^m x - p\|.$$

It follows that for all integers  $m \geq 1$ ,

$$\overline{\lim}_n \|T^n x - p\| = \overline{\lim}_n \|T^{n+m}x - p\| \leq \|T^m x - p\|$$

for  $\lim_n k_n = 1$ , which implies that  $\overline{\lim}_n \|T^n x - p\| \leq \overline{\lim}_m \|T^m x - p\|$  and, hence,  $\lim_n \|T^n x - p\|$  exists. To show that  $\{T^n x\}$  converges weakly to a fixed point of  $T$ , it suffices to show that

$$\omega_w(x) \subset F(T). \quad (3.2)$$

As a matter of fact, if (3.2) is proven and if  $p_i = w\text{-}\lim_j T^{n_j^{(i)}}(x)$  ( $i = 1, 2$ ) belong to  $\omega_w(x)$  and if  $p_1 \neq p_2$ , then Opial's condition of  $X$  implies

$$\begin{aligned}\lim_n \|T^n x - p_1\| &= \lim_j \|T^{n_j^{(1)}}(x) - p_1\| < \lim_j \|T^{n_j^{(1)}}(x) - p_2\| = \lim_j \|T^{n_j^{(2)}}(x) - p_2\| \\ &< \lim_j \|T^{n_j^{(2)}}(x) - p_1\| = \lim_n \|T^n x - p_1\|,\end{aligned}$$

a contradiction. Hence,  $\omega_w(x)$  must be a singleton. This amounts to saying the weak convergence of  $\{T^n x\}$ . Now let us prove (3.2). Let  $y = w\text{-}\lim T^n x$  be an arbitrary element of  $\omega_w(x)$ . By the weakly asymptotic regularity of  $T$  at  $x$ , we have for all integers  $m \geq 0$ ,

$$w\text{-}\lim_j T^{n_j+m} x = y.$$

Let

$$r_m = \overline{\lim}_j \|T^{n_j+m} x - y\|.$$

Then for all integers  $m, l \geq 0$ , by Opial's condition of  $X$ , we have

$$r_{m+l} = \overline{\lim}_j \|T^{n_j+m+l} x - y\| \leq \overline{\lim}_j \|T^{n_j+m+l} x - T^l y\| \leq k_l \cdot \overline{\lim}_j \|T^{n_j+m} x - y\| = k_l \cdot r_m.$$

It follows that for all integers  $m \geq 0$ ,

$$\overline{\lim}_l r_l = \overline{\lim}_l r_{l+m} \leq r_m,$$

which implies that the  $\lim_m r_m =: r$  exists. Now for all integers  $m, j \geq 0$ , we have

$$\begin{aligned} & \Phi(\|T^{n_j+2m} x - y\|) \\ &= \Phi(\|(T^{n_j+2m} x - T^m y) + (T^m y - y)\|) \\ &= \Phi(\|T^{n_j+2m} x - T^m y\|) + \int_0^1 \langle T^m y - y, J_\varphi(T^{n_j+2m} x - T^m y + t(T^m y - y)) \rangle dt \\ &\leq \Phi(k_m \|T^{n_j+m} x - y\|) + \int_0^1 \langle T^m y - y, J_\varphi(T^{n_j+2m} x - T^m y + t(T^m y - y)) \rangle dt. \end{aligned}$$

Taking the limit superior as  $j$  approaches the infinity, we get

$$\begin{aligned} \Phi(r_{2m}) &\leq \Phi(k_m r_m) + \int_0^1 \langle T^m y - y, J_\varphi(y - T^m y + t(T^m y - y)) \rangle dt \\ &= \Phi(k_m r_m) - \int_0^1 \|T^m y - y\| \varphi(t \|T^m y - y\|) dt \\ &= \Phi(k_m r_m) - \Phi(\|T^m y - y\|). \end{aligned}$$

Since  $\lim_m k_m = 1$  and  $\lim_m r_m$  exists, it follows that

$$\Phi(\|T^m y - y\|) \leq \Phi(k_m r_m) - \Phi(r_{2m}) \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

This implies  $T^m y \rightarrow y$  strongly and, hence,  $y$  is a fixed point of  $T$ . (3.2) is therefore proven. ■

#### 4. MALUTA'S CONSTANT

Let  $X$  be a Banach space. Then recall that Maluta's constant  $D(X)$  [17] of  $X$  is defined by

$$D(X) = \sup \left\{ \frac{\overline{\lim}_n d(x_{n+1}, \text{co}(x_1, \dots, x_n))}{\text{diam}(x_n)} \right\},$$

where the supremum is taken over all bounded nonconstant sequences  $\{x_n\}$  in  $X$ . It is known (cf. [17]) that if  $D(X) < 1$ , then  $X$  is reflexive and has normal structure and hence the fixed point property for nonexpansive mappings. However, it is not clear if  $D(X) < 1$  implies the fixed point property for asymptotically nonexpansive mappings. In this section we provide two partial answers to this question. For more details concerning some geometrical constants of Banach spaces, the reader is referred to [10, 17-20].

**THEOREM 4.** Suppose that  $X$  is a Banach space such that  $D(X) < 1$ , that  $C$  is a closed bounded convex subset of  $X$ , and that  $T: C \rightarrow C$  is an asymptotically nonexpansive mapping. Suppose, in addition, that  $T$  is weakly asymptotically regular on  $C$ , i.e.  $w\text{-}\lim_n (T^n x - T^{n+1} x) = 0$  for all  $x$  in  $C$ . Then  $T$  has a fixed point.

*Proof.* Let  $\mathcal{U}$  be a free ultrafilter on the set of positive integers. We then define a mapping  $S$  on  $C$  by

$$S(x) = w\text{-}\lim_{\mathcal{U}} T^n x, \quad x \in C.$$

Since  $C$  is weakly compact,  $S(x)$  is well defined for all  $x \in C$ . The asymptotic nonexpansiveness of  $T$  clearly implies that  $S$  is a nonexpansive mapping on  $C$ . Hence,  $S$  has a fixed point  $x \in C$ ; that is,

$$w\text{-}\lim_{\mathcal{U}} T^n x = x.$$

This yields a subsequence  $\{T^{n_i} x\}$  of  $\{T^n x\}$  converging weakly to  $x$ . Now we show that  $x$  is a fixed point of  $T$ . Take a real number  $q > 0$  small enough so that

$$(1 + q)^2 D(X) < 1.$$

Clearly we may assume that

$$k_{n_i} < 1 + q \quad \text{and} \quad k_{n_{i+1} - n_i} < 1 + q$$

for all integers  $i \geq 1$ . It then follows easily from the definition of  $D(X)$  that

$$\overline{\lim}_i \|T^{n_i} x - x\| \leq D(X) \cdot \text{diam}(\{T^{n_i} x\}).$$

However, for any fixed  $i > j$ , noting the fact that  $T^{n_i + (n_i - n_j)} x \rightarrow x$  weakly as  $t \rightarrow \infty$  and the weakly lower semicontinuity of the norm  $\|\cdot\|$ , we have

$$\begin{aligned} \|T^{n_i} x - T^{n_j} x\| &\leq k_{n_j} \|T^{n_i - n_j} x - x\| \\ &\leq (1 + q) \overline{\lim}_t \|T^{n_i - n_j} x - T^{n_i + (n_i - n_j)} x\| \\ &\leq (1 + q) k_{n_i - n_j} \overline{\lim}_t \|T^{n_i} x - x\| \\ &\leq (1 + q)^2 \cdot \overline{\lim}_i \|T^{n_i} x - x\|. \end{aligned}$$

We thus obtain

$$\overline{\lim}_i \|T^{n_i} x - x\| \leq (1 + q)^2 D(X) \overline{\lim}_i \|T^{n_i} x - x\|,$$

which implies that  $\overline{\lim}_i \|T^{n_i} x - x\| = 0$  for  $(1 + q)^2 D(X) < 1$ , which in turn implies that  $x$  is a fixed point of  $T$  because  $T$  is weakly asymptotically regular at  $x$ . ■

*Remark 1.* If  $T$  is nonexpansive, then it is well known (cf. [21]) that the averaged mappings  $T_a = aI + (1 - a)T$ , where  $a \in (0, 1)$  and  $I$  is the identity operator of  $X$ , are asymptotically regular on  $C$ , i.e.  $\lim_n \|T_a^n x - T_a^{n+1} x\| = 0$  for all  $x \in C$ . We do not know if this is valid for asymptotically nonexpansive mappings.

**THEOREM 5.** Suppose that  $X$  is a Banach space which is uniformly convex in every direction and for which  $D(X) < 1$  and that  $C$  is a closed bounded convex subset of  $X$ . Then, if  $T: C \rightarrow C$  is an asymptotically nonexpansive mapping,  $T$  has a fixed point.

*Proof.* Let  $K$  be the minimal subset of  $C$  constructed in the proof of theorem 3 (i) with respect to being nonempty, closed, convex and satisfying property (P). Then we have already known that for each  $x$  in  $K$ , the functional  $\overline{\lim}_n \|T^n x - y\|$  is a constant over  $y$  in  $C$ ; that is,

$$\overline{\lim}_n \|T^n x - y\| = r \quad \text{for all } y \in K.$$

But in a normed linear space that is uniformly convex in every direction, the asymptotic center of any bounded sequence relative to a weakly compact convex subset consists of exactly one point (cf. [22]), we conclude that  $K$  is a singleton, say  $\{z\}$ . Then in view of property (P), we see that the orbit  $\{T^n z\}$  of  $T$  at  $z$  converges weakly to  $z$  itself. Now proceeding exactly as the arguments in the proof of theorem 4, we obtain that  $z$  is a fixed point of  $T$ . ■

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