

Addendum to "Fixed Point Theorems for Uniformly Lipschitzian Mappings in L^p Spaces"

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In the proof of Lemma 2 in [1], one minor step was missing in the process of proving that $x(0+)$ is the solution of

$$(r-1)x^r + rx^{r-1} - 1 = 0 \tag{1}$$

in the interval $0 < x < 1$.

We have

$$\begin{aligned} & \lambda\mu \frac{1-\lambda^{r-1}}{1-\lambda} x(\mu)^r - \mu + r\lambda^{r-1}\mu x(\mu)^{r-1} - \binom{r}{2}\lambda^{r-2}\mu^2 x(\mu)^{r-2} + \dots \\ &= \lambda\mu \frac{1-\lambda^{r-1}}{1-\lambda} x(\mu)^r - \mu + r\lambda^{r-1}\mu x(\mu)^{r-1} + \lambda^r x(\mu)^r o(\alpha), \end{aligned}$$

where $\alpha = \frac{\mu}{\lambda x(\mu)}$. Thus $o(\alpha)/\mu \rightarrow 0$ as $\mu \rightarrow 0$ if one can prove that $x(\mu)$ is bounded away from 0. To this end, we shall prove that $x(\mu) \geq 1/4$ for $0 < \mu \leq 1/5$ (and hence for all $\mu \in (0, 1/2]$ since $x(\mu)$ is strictly increasing). For $0 < \mu \leq 1/5$, we have $\mu/\lambda \leq 1/4$; thus it suffices to show that $h(1/4) < 0$, or equivalently,

$$H(\mu) \equiv \lambda - \mu 4^r - (\lambda - 4\mu)^r < 0.$$

We have $H(0) = 0$ and $H'(\mu) = 5r(\lambda - 4\mu)^{r-1} - 4^r - 1 < 5r - 4^r - 1$. Simple calculus shows that $5r - 4^r - 1 < 0$. Thus $H(\mu)$ is strictly decreasing and $H(\mu) < 0$.

Also, the proof of Lemma 4 implies that $f(\mu) = \frac{g(\mu)}{\lambda\mu}$ is strictly decreasing in μ ; thus we have

$$2^{2-p} = f(1/2) < f(0+) = \frac{1 + \alpha^{p-1}}{(1 + \alpha)^{p-1}}.$$

Therefore we should have an improved bound for L in Theorem 2, namely, $L > (1 + 2^{2-p})^{1/p}$.

Remark 1 To show that $x(\mu)$ is bounded away from 0, we can also let x_0 be the unique solution of (1) in the interval $0 < x < 1$ and prove that $x(\mu) \geq x_0$. Let $F(\mu) = \lambda x_0^r - \mu - (\lambda x_0 - \mu)^r, 0 < \mu < 1/2$. It suffices to show that $F(\mu) < 0$ for $\mu > 0$. We have $F(0) = 0, F'(0) = (r-1)x_0^r + r x_0^{r-1} - 1 = 0$ and $F''(\mu) = -r(r-1)(\lambda x_0 - \mu)^{r-2}(x_0 + 1)^2 < 0$. It follows that $F(\mu) < 0$.

References

- [1] T. C. Lim, Fixed point theorems for uniformly Lipschitzian mappings in L^p spaces, *Nonlinear Analysis* **83** (1983), pp. 555 - 563.