Fact 1 Let \(a, b, c\) be reals, not all zeros. By multiplying \(Q(x, y) = ax^2 + 2bxy + cy^2 = 1\) if necessary, the quadratic equation \(Q(x, y) = ax^2 + 2bxy + cy^2 = 1\) represents an ellipse if \(ac - b^2 > 0\), a hyperbola if \(ac - b^2 < 0\), and two parallel lines (a degenerate parabola) if \(ac - b^2 = 0\).

Proof. The eigenvalues of
\[
A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}
\]
or \(-A\) are either both positive (in case \(ac - b^2 > 0\)), or one positive one negative (in case \(ac - b^2 < 0\)), or one 0 and the other positive (in case \(ac - b^2 = 0\)). This is so because the eigenvalues of \(A\) are
\[
a + c \pm \sqrt{(a + c)^2 - 4(ac - b^2)}.
\]
(We may assume that \(a + c \geq 0\).)

Fact 2 The equation \(u_{xy} = g(x, y, u, u_x, u_y)\) is transformed to \(u_{vv} - u_{ww} = h(v, w, u, u_v, u_w)\) with the coordinate change
\[
v = \frac{x + y}{2}, w = \frac{x - y}{2}.
\]
And it is transformed to \(u_{vv} + u_{ww} = h(v, w, u, u_v, u_w)\) with the coordinate change
\[
v = \frac{x + y}{2}, w = \frac{x - y}{2i}.
\]
Proof. Check that
\[4u_{xy} = u_{vv} - u_{ww}\]
in the first case, and
\[4u_{xy} = u_{vv} + u_{ww}\]
in the second case.
Theorem 1. Given a second order quasi-linear partial differential equation

\[ au_{xx} + 2bu_{xy} + cu_{yy} = g(x, y, u, u_x, u_y) \]

where \(a, b, c\) are functions of \((x, y)\) not all zeros, there exist \(\phi(x, y), \psi(x, y)\) such that relative to the new variables

\[ v = \phi(x, y), w = \psi(x, y) \]

the equation becomes

\[ u_{vv} + u_{ww} = h(v, w, u, u_v, u_w) \]

(in case \(ac - b^2 > 0\)), or

\[ u_{vw} = h(v, w, u, u_v, u_w) \]

(in case \(ac - b^2 < 0\)), or

\[ u_{vv} = h(v, w, u, u_v, u_w) \]

(in case \(ac - b^2 = 0\)).

Proof. We have

\[ u_{xx} = u_{vv}\phi_x^2 + 2u_{vw}\phi_x\psi_x + u_{ww}\psi_x^2 + \cdots, \]
\[ u_{xy} = u_{vv}\phi_y + u_{vw}(\phi_y + \phi_x\psi_y) + u_{ww}\psi_y + \cdots, \]
\[ u_{yy} = u_{vv}\phi_y^2 + 2u_{vw}\phi_y\psi_y + u_{ww}\psi_y^2 + \cdots, \]

where each \(\cdots\) does not contain second order partial derivatives of \(u\).

Then the original partial differential equation becomes

\[ \alpha u_{vv} + 2\beta u_{vw} + \gamma u_{ww} = h(v, w, u, u_v, u_w), \]

where

\[ \alpha = a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2, \]
\[ \beta = a\phi_x\psi_x + b(\phi_x\psi_y + \phi_y\psi_x) + c\phi_y\psi_y, \]
\[ \gamma = a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2. \]

We have the following relation:

\[ \alpha\gamma - \beta^2 = (ac - b^2)(\phi_x\psi_y - \phi_y\psi_x)^2. \quad (1) \]

Suppose \(a \neq 0\). If \(\lambda_1, \lambda_2\) are the two roots of the quadratic equation \(at^2 + 2bt + c = 0\) then \(\alpha\) and \(\gamma\) are factored to

\[ \alpha = a(\phi_x - \lambda_1\phi_y)(\phi_x - \lambda_2\phi_y), \]
\[ \gamma = a(\psi_x - \lambda_1\psi_y)(\psi_x - \lambda_2\psi_y). \]
Note: If \( a = 0 \) and \( c \neq 0 \), then the two roots \( \lambda_1, \lambda_2 \) are for the quadratic equation \( ct^2 + 2bt + a = 0 \), and the differential equations below should change to \( \frac{dy}{dx} + \lambda_1 = 0 \) and \( \frac{dy}{dx} + \lambda_2 = 0 \).

Also the roots of \( at^2 - 2bt + c = 0 \) (note the minus sign) are negative of that of \( at^2 + 2bt + c = 0 \). So the equations below can be written as \( \frac{dy}{dx} = \mu_i \) where \( \mu_i \) are the roots of \( at^2 - 2bt + c = 0 \).

Case (i): \( ac - b^2 < 0 \), the hyperbolic case: In this case, both \( \lambda_i \)'s are nonzero and of opposite sign. If we solve \( \frac{dy}{dx} + \lambda_1 = 0 \) and write its solution as \( \phi(x,y) = \text{const.} \), then \( \phi_x - \lambda_1 \phi_y = 0 \) (by implicit differentiation), and we have \( \alpha = 0 \).

Similarly if \( \psi(x,y) = \text{const.} \) is the solution of \( \frac{dy}{dx} + \lambda_2 = 0 \), then \( \gamma = 0 \). Note that by (1) \( \alpha \gamma - \beta^2 = -\beta^2 < 0 \) so \( \beta \neq 0 \). This means that if we make the coordinate transformation \( v = \phi(x,y) \) and \( w = \psi(x,y) \) then the original equation is transformed to

\[
u_{vw} = h(v, w, u, u_x, u_w)\]

Case (ii): \( ac - b^2 = 0 \), the parabolic case: In this case \( \lambda_1 = \lambda_2 \). If \( a = 0 \), then \( b = 0 \), and since \( a, b, c \) not all zeros by assumption, we have \( c \neq 0 \). Then the equation reduces to \( u_{yy} = (1/c)g(x,y,u,u_x,u_y) \), as required (with \( v = y, w = x \)). If \( a \neq 0 \), we can let \( v = x \) and \( w = \psi(x,y) \), where \( \psi \) is as in Case (i). Then \( \alpha = a \) and \( \gamma = 0 \). Therefore the original equation is transformed to the form

\[
u_{vw} = h(v, w, u, u_x, u_w)\]

Case (iii): \( ac - b^2 > 0 \), the elliptic case. Then both \( \lambda_1, \lambda_2 \) and complex and conjugate to each other. As in case (i), we solve \( \frac{dy}{dx} + \lambda_1 = 0 \) and write its solution as \( \phi(x,y) = \text{const.} \). Let \( \psi(x,y) \) be the conjugate of \( \phi(x,y) \). Then \( \psi(x,y) = \text{const.} \) is the solution of \( \frac{dy}{dx} + \lambda_2 = 0 \).

Let \( \xi = \phi(x,y) \) and \( \eta = \psi(x,y) \). Then as in case (i) the original equation becomes

\[
u_{\xi \eta} = h(\xi, \eta, u, u_\xi, u_\eta)\]

Now use Fact 2 above.

**Remark 1** The curves \( \phi(x,y) = \text{const.}, \psi(x,y) = \text{const.} \) are called the characteristic curves of the original equation.