

# On Moduli of k-Convexity

Teck-Cheong Lim  
 Department of Mathematical Sciences  
 George Mason University  
 4400 University Drive  
 Fairfax, VA 22030  
 U.S.A.

October 11, 1999

## Abstract

In this paper, we establish the continuity of some moduli of k-convexity.

Let  $X$  be a Banach space. We shall denote  $X^*$  the dual space of  $X$  and  $B_X$  the unit ball of  $X$ . Several moduli of convexity for the norm of  $X$  have been defined; the last two definitions in the following are valid for spaces having dimension  $\geq k$ :

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in B_X, \|x - y\| \geq \epsilon\right\} \quad [2]$$

$$\delta_X^{(k)}(\epsilon) = \inf\left\{1 - \frac{\|x_1 + \cdots + x_{k+1}\|}{k+1} : x_1, \dots, x_{k+1} \in B_X, A(x_1, \dots, x_{k+1}) \geq \epsilon\right\} \quad [10]$$

$$\Delta_X^{(k)}(\epsilon) = \inf_{\|x\|=1} \inf_{\substack{Y \subset X \\ \dim(Y)=k}} \sup_{\substack{\|y\|=1 \\ y \in Y}} \{\|x + \epsilon y\| - 1\} \quad [9],$$

where

$$A(x_1, \dots, x_{k+1}) = \frac{1}{k!} \sup\left\{ \begin{vmatrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+1}) \\ \vdots & \cdots & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+1}) \end{vmatrix} : f_1, \dots, f_k \in B_{X^*} \right\}.$$

Evidently, by subtracting the first column from the other columns, the determinant can be replaced by

$$\begin{vmatrix} f_1(x_2 - x_1) & \cdots & f_1(x_{k+1} - x_1) \\ \vdots & \cdots & \vdots \\ f_k(x_2 - x_1) & \cdots & f_k(x_{k+1} - x_1) \end{vmatrix}. \quad (1)$$

Also  $A(x_1, \dots, x_{k+1})$  can be thought of as the "volume" of the convex hull of  $x_1, \dots, x_{k+1}$  since that is the case in Euclidean spaces.

$X$  is called uniformly convex if  $\delta_X(\epsilon) > 0$  for  $\epsilon > 0$  and  $k$ -uniformly convex if  $\delta_X^{(k)}(\epsilon) > 0$  for  $\epsilon > 0$ . Note that  $\delta_X(\epsilon) = \delta_X^{(1)}(\epsilon)$ ; so 1-uniform convexity coincides with uniform convexity. Lin [8] proved that  $\Delta_X^{(k)}(\epsilon) > 0$  for  $\epsilon > 0$  is equivalent to  $k$ -uniform convexity. Gurarii [5] proved that  $\delta_X(\epsilon)$  is continuous on  $[0, 2)$  and there exist spaces of which  $\delta_X(\epsilon) = 0$  for  $0 \leq \epsilon < 2$  and  $\delta_X(2) = 1$ . The continuity problem of  $\delta_X^{(k)}$  was mentioned in Kirk [6]. Let  $\mu_X^{(k)} = \sup\{A(x_1, \dots, x_{k+1}) : x_1, \dots, x_{k+1} \in B_X\}$ . Note that  $\mu_X^{(1)} = 2$ . In this note we prove that  $\delta_X^{(k)}(\epsilon)$  is continuous on  $[0, \mu_X^{(k)})$ . It is quite evident that  $\Delta_X^{(k)}(\epsilon)$  satisfy the Lipschitz condition with constant 1.

**Definition 1** Let  $k \geq 1$  and  $0 \leq a < b \leq \infty$ . A function  $f(\epsilon)$  on  $(a, b)$  is called  $k$ -convex if

$$f((\lambda\epsilon_2^{1/k} + (1-\lambda)\epsilon_1^{1/k})^k) \leq \lambda f(\epsilon_2) + (1-\lambda)f(\epsilon_1)$$

for every  $\epsilon_1, \epsilon_2 \in (a, b), 0 \leq \lambda \leq 1$ .

Obviously 1-convexity is simply the ordinary convexity.

**Lemma 2** Let  $0 \leq a < b \leq \infty$  and let  $f$  be a nondecreasing  $k$ -convex function on  $(a, b)$  with  $M = \sup_{a < x < y < b} (f(y) - f(x)) < \infty$ . Let  $\epsilon_1 < \epsilon_2, \epsilon_1, \epsilon_2 \in (a, b)$ . Then

$$\frac{f(c) - f(\epsilon_1)}{c - \epsilon_1} \leq \frac{M}{k(\epsilon_2^{1/k} - \epsilon_1^{1/k})\epsilon_1^{1-1/k}}$$

for every  $\epsilon_1 < c < \epsilon_2$ .

**Proof.** Let  $z(x), \epsilon_1 \leq x \leq \epsilon_2$  be the function whose graph is defined by

$$\begin{cases} x &= (\lambda\epsilon_2^{1/k} + (1-\lambda)\epsilon_1^{1/k})^k \\ y &= \lambda f(\epsilon_2) + (1-\lambda)f(\epsilon_1) \end{cases} \quad 0 \leq \lambda \leq 1.$$

By direct computation, we have

$$\begin{aligned} z'(x) &= \frac{f(\epsilon_2) - f(\epsilon_1)}{k(\epsilon_2^{1/k} - \epsilon_1^{1/k})(\lambda\epsilon_2^{1/k} + (1-\lambda)\epsilon_1^{1/k})^{k-1}} \\ &\leq \frac{M}{k(\epsilon_2^{1/k} - \epsilon_1^{1/k})\epsilon_1^{1-1/k}}. \end{aligned}$$

If  $\epsilon_1 < c < \epsilon_2$ , then by the  $k$ -convexity of  $f$  and Mean-Value Theorem,

$$\frac{f(c) - f(\epsilon_1)}{c - \epsilon_1} \leq \frac{z(c) - z(\epsilon_1)}{c - \epsilon_1} = z'(\psi) \leq \frac{M}{k(\epsilon_2^{1/k} - \epsilon_1^{1/k})\epsilon_1^{1-1/k}}$$

The inequality in the following lemma is a consequence of a more general result proved in Bernal-Sullivan [1].

**Lemma 3** Let  $X$  be a Banach space and  $x_1, \dots, x_{k+1} \in X$ . Then

$$A(x_1, \dots, x_{k+1}) \leq \frac{1}{k!} k^{k/2} \|x_2 - x_1\| \cdots \|x_{k+1} - x_1\|.$$

**Proof.** Hadamard inequality says that if  $r_1, r_2, \dots, r_k$  are the rows (or columns) of a  $k \times k$  matrix, then

$$\det(r_1, r_2, \dots, r_k) \leq \|r_1\|_2 \|r_2\|_2 \cdots \|r_k\|_2.$$

Here  $\|\cdot\|_2$  denotes the Euclidean norm in  $\mathbf{R}^k$ . Since the Euclidean norm of the  $j$ -th column of the determinant in (1) is  $\leq k^{1/2} \|x_{j-1} - x_1\|$ , the inequality follows.

The inequality in the next theorem for the case  $k = 1$  improves the one obtained in [5]. The general idea is similar to that in Goebel [3]. However, the reader should be aware that the assertion of Lemma 1 in that paper (that  $\delta(\epsilon)$  is convex) is incorrect; a counterexample can be found in [7] or [4].

**Theorem 4** Let  $X$  be a Banach space. Then

$$\frac{\delta_X^{(k)}(c) - \delta_X^{(k)}(\epsilon_1)}{c - \epsilon_1} \leq \frac{1}{k(\epsilon_2^{1/k} - \epsilon_1^{1/k})\epsilon_1^{1-1/k}}. \quad (2)$$

for every  $0 < \epsilon_1 < c < \epsilon_2 < \mu_X^{(k)}$ .

**Proof.** For simplicity, in the following we will consider  $k = 2$  and will indicate how to generalize to general  $k$ . Note that if  $A(x_1, x_2, x_3) > 0$ , then  $x_2 - x_1$  and  $x_3 - x_1$  are linearly independent.

For unit vectors  $u, u_{21}, u_{31}$  and  $u_{32}$  in  $X$ , with  $\{u_{21}, u_{31}\}$  linearly independent, consider the set

$$N(u, u_{21}, u_{31}, u_{32}; \epsilon) = \{(x_1, x_2, x_3) \in X^3 : x_1 + x_2 + x_3 = \lambda u, x_2 - x_1 = \lambda_{21} u_{21}, x_3 - x_1 = \lambda_{31} u_{31}, \\ x_3 - x_2 = \lambda_{32} u_{32} \text{ for some } \lambda, \lambda_{21}, \lambda_{31}, \lambda_{32} \geq 0 \text{ and } A(x_1, x_2, x_3) \geq \epsilon\}$$

and define

$$\delta(u, u_{21}, u_{31}, u_{32}; \epsilon) = \inf \left\{ 1 - \frac{\|x_1 + x_2 + x_3\|}{3} : (x_1, x_2, x_3) \in N(u, u_{21}, u_{31}, u_{32}; \epsilon) \right\}.$$

Obviously,  $\delta(u, u_{21}, u_{31}, u_{32}; \epsilon)$  is nondecreasing and has values in  $[0, 1]$ .

If  $(x_1, x_2, x_3) \in N(u, u_{21}, u_{31}, u_{32}; \epsilon_1)$ ,  $(y_1, y_2, y_3) \in N(u, u_{21}, u_{31}, u_{32}; \epsilon_2)$  and

$$\begin{aligned} x_1 + x_2 + x_3 &= \lambda u, & x_2 - x_1 &= \lambda_{21} u_{21}, & x_3 - x_1 &= \lambda_{31} u_{31}, & x_3 - x_2 &= \lambda_{32} u_{32} \\ y_1 + y_2 + y_3 &= \alpha u, & y_2 - y_1 &= \alpha_{21} u_{21}, & y_3 - y_1 &= \alpha_{31} u_{31}, & y_3 - y_2 &= \alpha_{32} u_{32} \end{aligned}$$

for some  $\lambda, \lambda_{ij}, \alpha, \alpha_{ij} \geq 0$ , then by linear independence of  $\{u_{21}, u_{31}\}$ , there exists  $c \geq 0$  such that

$$\alpha_{21} = c\lambda_{21}, \alpha_{31} = c\lambda_{31}, \alpha_{32} = c\lambda_{32}$$

Indeed,  $\lambda_{32}u_{32} = x_3 - x_2 = (x_3 - x_1) - (x_2 - x_1) = \lambda_{31}u_{31} - \lambda_{21}u_{21}$  and  $\alpha_{32}u_{32} = \alpha_{31}u_{31} - \alpha_{21}u_{21}$  imply

$$(\alpha_{32}\lambda_{31} - \lambda_{32}\alpha_{31})u_{31} - (\alpha_{32}\lambda_{21} - \lambda_{32}\alpha_{21})u_{21} = 0$$

from which it follows  $\frac{\alpha_{31}}{\lambda_{31}} = \frac{\alpha_{32}}{\lambda_{32}} = \frac{\alpha_{21}}{\lambda_{21}}$ .

Let

$$C(u_{21}, u_{31}) = \sup\left\{ \begin{array}{cc} f_1(u_{21}) & f_1(u_{31}) \\ f_2(u_{21}) & f_2(u_{31}) \end{array} \middle| : f_1, f_2 \in B_{X^*} \right\}$$

Then  $A(x_1, x_2, x_3) = \lambda_{21}\lambda_{31}C(u_{21}, u_{31}) \geq \epsilon_1$  and

$A(y_1, y_2, y_3) = c^2\lambda_{21}\lambda_{31}C(u_{21}, u_{31}) \geq \epsilon_2$ .

For  $0 \leq \zeta \leq 1$ , let  $z_i = \zeta x_i + (1 - \zeta)y_i, i = 1, 2, 3$ . Then

$$z_2 - z_1 = (\zeta\lambda_{21} + (1 - \zeta)c\lambda_{21})u_{21} = (\zeta + (1 - \zeta)c)\lambda_{21}u_{21},$$

$$z_3 - z_1 = (\zeta + (1 - \zeta)c)\lambda_{31}u_{31},$$

$$z_3 - z_2 = (\zeta + (1 - \zeta)c)\lambda_{32}u_{32},$$

$$z_1 + z_2 + z_3 = (\zeta\lambda + (1 - \zeta)\alpha)u,$$

$$A(z_1, z_2, z_3) = (\zeta + (1 - \zeta)c)^2\lambda_{21}\lambda_{31}C(u_{21}, u_{31}) \geq (\zeta\epsilon_1^{1/2} + (1 - \zeta)\epsilon_2^{1/2})^2$$

and

$$\begin{aligned} 1 - \frac{\|z_1 + z_2 + z_3\|}{3} &= 1 - \frac{\|\zeta(x_1 + x_2 + x_3) + (1 - \zeta)(y_1 + y_2 + y_3)\|}{3} \\ &= 1 - \frac{\|\zeta\lambda u + (1 - \zeta)\alpha u\|}{3} \\ &= 1 - \frac{\zeta\lambda + (1 - \zeta)\alpha}{3} \\ &= \zeta\left(1 - \frac{\lambda}{3}\right) + (1 - \zeta)\left(1 - \frac{\alpha}{3}\right) \\ &= \zeta\left(1 - \frac{\|x_1 + x_2 + x_3\|}{3}\right) + (1 - \zeta)\left(1 - \frac{\|y_1 + y_2 + y_3\|}{3}\right) \end{aligned}$$

Hence

$$\begin{aligned} & \delta(u, u_{21}, u_{31}, u_{32}; (\zeta\epsilon_1^{1/2} + (1-\zeta)\epsilon_2^{1/2})^2) \\ \leq & \zeta\delta(u, u_{21}, u_{31}, u_{32}; \epsilon_1) + (1-\zeta)\delta(u, u_{21}, u_{31}, u_{32}; \epsilon_2) \end{aligned}$$

Since

$$\delta_X^{(2)}(\epsilon) = \inf\{\delta(u, u_{21}, u_{31}, u_{32}; \epsilon) : \|u\| = \|u_{21}\| = \|u_{31}\| = \|u_{32}\| = 1, \\ \{u_{21}, u_{31}\} \text{ linearly independent}\},$$

and the inequality in Lemma 2 is preserved under passing to infimum, inequality (2) for  $k = 2$  follows.

For general  $k$ , we have  $\binom{k+1}{2} + 1$  unit vectors  $u, u_{21}, \dots$  and the proof is similar to the one above.

**Corollary 5** *Let  $X$  be a Banach space. Then  $\delta_X^{(k)}(\epsilon)$  is continuous on  $[0, \mu_X^{(k)})$ .*

**Proof.**

Take  $\|x_1\| = 1$  and  $x_2, \dots, x_{k+1}$  in a small ball centered at  $x_1$ . Then by Lemma 3  $A(x_1, \dots, x_{k+1})$  is small. Since  $1 - \frac{\|x_1 + \dots + x_{k+1}\|}{k+1}$  is close to 0, we see that  $\delta_X^{(k)}(\epsilon)$  is continuous at 0.

Continuity of  $\delta_X^{(k)}(\epsilon)$  on  $(0, \mu_X^{(k)})$  follows immediately from the inequality (2).

*Acknowledgment:* The author wish to thank the referee for his comments that led to a better formulation of the results in this paper.

## References

- [1] J. Bernal and F. Sullivan, Multi-dimensional volumes, super-reflexivity and normal structure in Banach spaces, *Illinois J. Math.* **27**(1983), pp. 501-513.
- [2] A. Clarkson, Uniformly Convex spaces, *Trans. Amer. Math. Soc.* **40** (1936), pp. 396 - 414.
- [3] K. Goebel, Convexity of balls and fixed point theorems for mappings with a nonexpansive square, *Compositio Math.* **22** (1970), pp. 269-274.
- [4] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*. Cambridge Studies in Advanced Mathematics, **28**. Cambridge University Press, Cambridge, 1990. viii+244 pp.
- [5] V. I. Gurarii, On the differential properties of the modulus of convexity in a Banach space (in Russian), *Mat. Issled.* **2** (1967), pp. 141-148.

- [6] W. A. Kirk, Nonexpansive mappings in product spaces, set-valued mappings and k-uniform rotundity, *Nonlinear Functional Analysis and Applications* F.E. Browder, ed., Proc. symp. Pure Math. **45** pt. 2, Amer. Math. Soc., R.I., 1986, pp. 51-64.
- [7] T. C. Lim, *The asymptotic center and fixed point theorems for nonexpansive mappings*. Ph. D. Thesis, Dalhousie University, 1974.
- [8] P. K. Lin, k-Uniform Rotundity is equivalent to k-uniform convexity, *J. Math. Anal. and Appl.* **132** (1988), pp. 349-355.
- [9] V. D. Milman, Geometric theory of Banach spaces II, *Vspehi Mat. Nauk.* **26** (1971), pp. 79-163. Translated in Russian Math. Surveys.
- [10] F. Sullivan, A generalization of uniformly rotund Banach spaces, *Canadian J. Math.*, **31** (1979), pp. 628-636.