Proof of Descartes’s Rule of Signs

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The following proof, based on a paper by Wang, uses some facts from calculus.

Definition 1 Let \( a \) be a real zero of a polynomial \( P(x) \). We say that the graph of \( P \) crosses the \( x \)-axis at \( a \) if there is an \( \epsilon > 0 \) such that \( P \) assumes different signs on \( (a-\epsilon,a) \) and \( (a,a+\epsilon) \); i.e. either \( P(x) > 0 \) on \( (a-\epsilon,a) \) and \( P(x) < 0 \) on \( (a,a+\epsilon) \), or \( P(x) < 0 \) on \( (a-\epsilon,a) \) and \( P(x) > 0 \) on \( (a,a+\epsilon) \).

Fact 1 Let \( a \) be a real zero of a polynomial \( P(x) \). The graph of \( P \) crosses the \( x \)-axis at \( a \) if and only if the multiplicity of \( a \) is odd.

Proof. Suppose that the multiplicity of \( a \) is \( m \), \( m \) being an odd positive integer. Then \( P(x) = (x-a)^m Q(x) \), where \( Q \) is a polynomial with \( Q(a) \neq 0 \). Assume that \( Q(a) > 0 \). By continuity of \( Q \), there exists \( \epsilon > 0 \) such that \( Q(x) > 0 \) for all \( x \in (a-\epsilon,a+\epsilon) \). It follows that \( P(x) < 0 \) on \( (a-\epsilon,a) \) and \( P(x) > 0 \) on \( (a,a+\epsilon) \). Similarly if \( Q(a) < 0 \), then \( P(x) > 0 \) on \( (a-\epsilon,a) \) and \( P(x) < 0 \) on \( (a,a+\epsilon) \) for some \( \epsilon > 0 \).

The same argument shows that if \( m \) is an even positive integer, then \( P \) assumes the same sign on \( (a-\epsilon,a) \) and \( (a,a+\epsilon) \) for some \( \epsilon > 0 \), proving that \( P \) does not cross the \( x \)-axis at \( a \). Q.E.D.

Fact 2 A polynomial of degree \( n, n \geq 1 \) can have at most \( n \) real zeros, counting multiplicities.

Fact 3 Let \( P(x) = a_0 + a_1 x + \cdots + a_n x^n, n \geq 1, a_n \neq 0 \). Then \( \lim_{x \to \infty} P(x) = \infty \) if \( a_n > 0 \), and \( \lim_{x \to -\infty} P(x) = -\infty \) if \( a_n < 0 \).

Proposition 1 Let \( P(x) = a_0 x^{b_0} + \cdots + a_n x^{b_n}, \) where \( a_i, i = 0, \cdots, n \) are nonzero real numbers, and \( 0 \leq b_0 < b_1 < \cdots < b_n \) are integers. Then \( P(x) \) has an even number of positive zeros, counting multiplicities, if and only if \( a_0 a_n > 0 \).
Proof.
Since $P$ has the same number of positive zeros as $a_0 + a_1 x^{c_1} + \cdots + a_n x^{c_n}$, where $c_i = b_i - b_0$, we may assume that $b_0 = 0$. $a_0 a_n > 0$ implies that $P(0) = a_0$ and $\lim_{x \to \infty} P(x)$ have the same sign. It follows that $P$ can cross the positive $x$-axis an even number of times, each time corresponding to a zero of odd multiplicity by Fact 1. $P$ may have other positive zeros of even multiplicity. Therefore $P$ has an even number of positive zeros. On the other hand, if $a_0 a_n < 0$, the same argument proves that $P$ has an odd number of positive zeros. Q.E.D.

Fact 4 Suppose $a$ is a real zero of polynomial $P(x)$ of multiplicity $m$, then $a$ is a zero of $P'(x)$, the derivative of $P$, of multiplicity $m - 1$ if $m \geq 2$; not a zero of $P'$ if $m = 1$.

Proposition 2 Let $z(P), z(P')$ denote the number of positive zeros of $P$ and $P'$ respectively. Then

$$z(P') \geq z(P) - 1$$

Proof.
Suppose $z_0 < z_1 < \cdots < z_k$ are the positive zeros of $P$ of multiplicities $m_0, \cdots, m_k$ respectively. By Rolle’s theorem $P'$ has at least one zero strictly between each consecutive $z_i$’s. It follows from Fact 4 that

$$z(P') \geq (m_0 - 1) + \cdots + (m_k - 1) + k = z(P) - 1$$

Notation 1 Let $P(x) = a_0 x^{b_0} + \cdots + a_n x^{b_n}$, where $a_i, i = 0, \ldots, n$ are nonzero real numbers, and $0 \leq b_0 < b_1 < \cdots < b_n$ are integers. $v(P)$ will denote the number of sign changes in the sequence $a_0, \ldots, a_n$.

Theorem 1 (Descartes’s Rule of Signs) Let $P(x) = a_0 x^{b_0} + \cdots + a_n x^{b_n}$, where $a_i, i = 0, \ldots, n$ are nonzero real numbers, and $0 \leq b_0 < b_1 < \cdots < b_n$ are integers. The number of positive zeros of $P$, counting multiplicities, is either equal to $v(P)$ or less than that by an even number.

Proof.
The theorem is evidently true for $b_n = 1$. Assume that it is true for polynomials of degree less than $b_n$. Consider two cases:

Case 1: $a_0 a_1 > 0$. Then $v(P) = v(P')$. By induction hypothesis $z(P') \leq v(P')$ and $z(P') = v(P') \mod 2$. By Proposition 1, $z(P) = z(P') \mod 2$. So $z(P) = v(P) \mod 2$. By Proposition 2,

$$z(P) \leq z(P') + 1 \leq v(P') + 1 = v(P) + 1$$

This together with $z(P) = v(P) \mod 2$ yields $z(P) \leq v(P)$ and the conclusion of the theorem.

Case 2: $a_0 a_1 < 0$. Then $v(P) = v(P') + 1$. By induction hypothesis $z(P') \leq v(P')$ and $z(P') = v(P') \mod 2$. By Proposition 1, $z(P) \neq z(P') \mod 2$, which together with $v(P) \neq v(P') \mod 2$ and $v(P') = z(P') \mod 2$ yields $z(P) = v(P) \mod 2$. By Proposition 2,

$$z(P) \leq z(P') + 1 \leq v(P') + 1 = v(P)$$
This completes the proof. Q.E.D.

**Corollary 1** Let $P(x) = a_0x^{b_0} + \cdots + a_nx^{b_n}$, where $a_i, i = 0, \ldots, n$ are nonzero real numbers, and $0 \leq b_0 < b_1 < \cdots < b_n$ are integers. The number of negative zeros of $P$, counting multiplicities, is either equal to $v(P^*)$ or less than that by an even number, where $P^*(x) = P(-x)$.

**Proof.**
a is a negative zero of $P(x)$ with multiplicity $m$ if and only if $-a$ is a positive zero of $P(-x)$ of the same multiplicity.

**References**