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A FIXED POINT THEOREM FOR FAMILIES OF NONEXPANSIVE MAPPINGS

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In this paper it is proved that if K is a weakly compact convex subset of a Banach space and if K has normal structure, then any family of commuting nonexpansive mappings on K into itself admits a common fixed point.

Kirk [5] first proved that if K is a nonempty weakly compact convex subset of a Banach space and if K has normal structure [3], then every nonexpansive mapping $T: K \to K$ has a fixed point. Later, Belluce and Kirk [1] extend this theorem by showing that any finite family of commuting nonexpansive self-mappings of such a set K always has a common fixed point. Their attempts to prove a common fixed point theorem for arbitrary families resulted in the need for a strengthening of normal structure called complete normal structure (see [2]). Since then the problem of whether their theorem is true for arbitrary families under the normal structure setting remained unsolved. In the present paper, we shall solve this problem by giving an affirmative answer. A technique used in the proof is the notion of asymptotic center which was first considered by Edelstein [4] to obtain a strong version of fixed point theorem in uniformly convex Banach spaces.

Throughout this paper, we shall denote the diameter of a set A of X by $\delta(A)$.

DEFINITION 1. Let $\{x_\alpha\}_{\alpha<\gamma}$ be a bounded net, ordered by ordinals less than γ, in a convex set C of a Banach space X, where γ is an ordinal $\geqq 1$. For every $x \in C$ and every ordinal $\beta < \gamma$, define

$$r_\beta(x) = \sup\{\|x - x_\alpha\|: \alpha \geqq \beta\}$$

$$r(x) = \inf\{r_\delta(x): \delta < \gamma\} = \overline{\lim}\, \|x - x_\alpha\|$$

and

$$r = \inf\{r(x): x \in C\}\ .$$

The set $\{x \in C: r(x) = r\}$ (the number r) will be called the asymptotic center (asymptotic radius) of $\{x_\alpha\}_{\alpha<\gamma}$ in C.

REMARK. Our main concern in this paper is the case where $\gamma = \aleph_\delta$ for some $\delta \geqq 0$.

Some basic properties of $r(x)$ and asymptotic center:

1. For each $x \in C$, $\{r_\beta(x)\}_{\beta<\gamma}$ is a decreasing net with limit $r(x)$.
2. $r(x) = 0$ if and only if $x_\alpha \to x$.
3. $|r(x) - r(y)| \leqq \|x - y\|$ for every $x, y \in C$. This follows from (1) and the fact that $|r_\beta(x) - r_\beta(y)| \leqq \|x - y\|$ for every $\beta < \gamma$.
4. $r(x)$ is a continuous convex function on C. This follows from (3), (1) and the fact that $r_\beta(x)$ are convex functions on C for all $\beta < \gamma$.
5. $\{x \in C: r(x) = r\}$ is a closed convex subset of C. This follows from (4).
6. If C is weakly compact convex, then $\{x \in C: r(x) = r\}$ is nonempty. This follows from (4), the equality $\{x \in C: r(x) = r\} = \bigcap_{n=1}^{\infty} \{x \in C: r(x) \leqq r + 1/n\}$ and that closed convex subsets of weakly compact set are weakly compact.

DEFINITION 2. A convex subset C of a Banach space is said to have normal structure [3] if every bounded convex subset D of C with $|D| > 1$ contains a point x such that $\sup \{\|x - y\|: y \in D\} < \delta(D)$. A convex subset C of a Banach space is said to have $\aleph_\delta$-normal structure, $\delta \geqq 0$, if for every nonconstant bounded net $\{x_\alpha\}_{\alpha<\aleph_\delta}$ in C, the asymptotic center of $\{x_\alpha\}_{\alpha<\aleph_\delta}$ in $\mathrm{Co}\,(x_\alpha)_{\alpha<\aleph_\delta}$ is a proper subset of $\mathrm{Co}\,(x_\alpha)_{\alpha<\aleph_\delta}$; in case that C has $\aleph_\delta$-normal structure for every $\delta \geqq 0$, we say that C has asymptotic normal structure.

In their original paper [3], Brodskii and Milman characterized normal structure as follows: A convex subset of a Banach space has normal structure if and only if it contains no diametral sequences. (A diametral sequence is a nonconstant bounded sequence $\{x_n\}_{n=1}^{\infty}$ such that $d(x_{n+1}, \mathrm{Co}\,(x_1, \cdots, x_n)) \to \delta(\{x_n\}_{n=1}^{\infty})$.) The following lemma is a simple variation of the above characterization.

LEMMA 1. *A convex subset C of a Banach space has normal structure if and only if it does not contain a sequence $\{x_n\}$ such that for some $c > 0$, $\|x_n - x_m\| \leqq c$, $\|x_{n+1} - \bar{x}_n\| \geqq c - 1/n^2$ for all $n \geqq 1$, $m \geqq 1$, where $\bar{x}_n = 1/n \sum_{i=1}^{n} x_i$.*

Proof. If C contains a bounded convex subset such that $|D| > 1$ and $\sup \{\|x - y\|: y \in D\} = \delta(D)$ for every $x \in D$, then it is easy to choose, by induction, a nonconstant sequence $\{x_n\} \subseteqq D$ satisfying the condition in the lemma with $c = \delta(D)$. On the other hand, assume that $\{x_n\} \subseteqq C$ is a sequence satisfying the condition in the lemma. If $x \in \mathrm{Co}\,(x_1, \cdots, x_n)$, it is not difficult to show that $x = \lambda_1 \bar{x}_n + \lambda_2 x_{i_2} + \cdots + \lambda_n x_{i_n}$ for some $i_2, \cdots, i_n \in \{1, \cdots, n\}$ and λ_i, $1 \leqq i \leqq n$, with $\sum_{i=1}^{n} \lambda_i = 1$, $0 < \lambda_1 \leqq n$, and $\lambda_j \leqq 0$ for $2 \leqq j \leqq n$. It follows that $c \geqq \|x_{n+1} - x\| \geqq c - 1/n$ for every $n \geqq 1$ and every $x \in \mathrm{Co}\,(x_1, \cdots, x_n)$. Hence $d(x_{n+1}, \mathrm{Co}\,(x_1, \cdots, x_n)) \to c$ and c is necessarily equal to $\delta(\{x_n\})$.

In what follows, we shall use $r_\beta(x)$, $r(x)$ and r as is defined in Definition 1 without referring to the original net $\{x_\alpha\}_{\alpha<\Gamma}$ if no ambiguity can arise.

PROPOSITION 1. *A convex subset C of a Banach space has normal structure if and only if it has $\aleph_0$-normal structure.*

Proof. Suppose C has $\aleph_0$-normal structure and suppose on the contrary that it contains a diametral sequence $\{u_i\}_{i=1}^\infty$ with diameter δ. For every $x \in \mathrm{Co}\,(u_i)_{i=1}^\infty$, there is an integer N such that $x \in \mathrm{Co}\,(u_1, \cdots, u_m)$ for all $m \geqq N$. Since $d(u_{i+1}, \mathrm{Co}\,(u_1, \cdots, u_i)) \to \delta$ as $i \to \infty$, we have $r(x) = \overline{\lim}_i \| u_i - x \| = \delta$ for every $x \in \mathrm{Co}\,(u_i)_{i=1}^\infty$. This implies that $r = \delta$ and the asymptotic center of $\{u_i\}_{i=1}^\infty$ in $\mathrm{Co}\,(u_i)_{i=1}^\infty$ in $\mathrm{Co}\,(u_i)_{i=1}^\infty$ itself—a contradiction. Hence C has normal structure by Brodskii-Milman's characterization.

Suppose now that C does not have $\aleph_0$-normal structure. Then there exists a nonconstant bounded sequence $\{u_i\}_{i=1}^\infty$ such that $r(x) = r$ for every $x \in \mathrm{Co}\,(u_i)_{i=1}^\infty$. By the basic property (2) following Definition 1, we have $r > 0$. Denoto $\mathrm{Co}\,(u_i)_{i=1}^\infty$ by D.

Let x_1 be an arbitrary point in C. Since $r(x_1) = r$, there exists $x_2 \in \{u_i\}_{i=1}^\infty$ such that $\| x_1 - x_2 \| \geqq r - 1$; move x_2 towards x_1 along the line segment joining x_1 and x_2 if necessary, we may assume that $\| x_1 - x_2 \| \leqq r$. Suppose now that $\{x_1, \cdots, x_n\} \subsetneqq D$, $n \geqq 2$, has been chosen such that

$$(1) \qquad \| x_i - x_j \| \leqq r \qquad (1 \leqq i \leqq n, 1 \leqq j \leqq n)$$

and

$$\| x_n - \bar{x}_{n-1} \| \geqq r - \frac{1}{(n-1)^2}$$

where $\bar{x}_{n-1} = \sum_{i=1}^{n-1} x_i$. We proceed to choose $x_{n+1} \in D$ as follows: Let m be an integer such that

$$(2) \qquad r_m(x_i) \leqq r + \frac{1}{n^2(n+1)} \qquad (1 \leqq i \leqq n)$$

and

$$(3) \qquad r_m(\bar{x}_n) \leqq r + \frac{1}{n^2(n+1)} .$$

Choose $z_0 \in \{u_m, u_{m+1}, \cdots\}$ so that

$$(4) \qquad r_m(\bar{x}_n) - \| z_0 - \bar{x}_n \| \leqq \frac{1}{n^2(n+1)} .$$

Since $\max(\|z_0 - \bar{x}_n\|, r) \leqq r_m(\bar{x}_n)$, we have, from (3) and (4),

$$\left| \|z_0 - \bar{x}_n\| - r \right| \leqq \frac{1}{n^2(n+1)}. \tag{5}$$

Let $\{z_1, \cdots, z_n\}$ be defined recursively by the formulae

$$z_i = t_i x_i + (1 - t_i) z_{i-1} \qquad (1 \leqq i \leqq n) \tag{6}$$

where

$$t_i = \max\left(\frac{\|x_i - z_{i-1}\| - r}{\|x_i - z_{i-1}\|}, 0\right)$$

for each $1 \leqq i \leqq n$. We then define $x_{n+1} = z_n$.

Geometrically, x_{n+1} is obtained by first moving z_0 along the line segment joining x_1 and z_0 so that its distance with x_1 becomes r (keeping z_0 unmoved if its distance with x_1 is already not greater than r) and then moving the resulting point towards x_2 in the same manner and continuing so on, up to n times.

It is geometrically clear and can be easily proved that $\|x_{n+1} - x_i\| \leqq r$ for all $1 \leqq i \leqq n$. By making use of (1), (2), (6) and the fact that $z_0 \in \{u_m, u_{m+1}, \cdots\}$, one can easily show that

$$\|x_i - z_{i-1}\| \leqq r + \frac{1}{n^2(n+1)} \tag{7}$$

for all $1 \leqq i \leqq n$. Since

$$\|z_i - z_{i-1}\| = t_i \|x_i - z_{i-1}\| \leqq \max(\|x_i - z_{i-1}\| - r, 0)$$

by (6), we have, by (7),

$$\|x_{n+1} - z_0\| \leqq \|z_0 - z_1\| + \cdots + \|z_{n-1} - z_n\| \leqq \frac{n}{n^2(n+1)}$$

and therefore

$$\|x_{n+1} - \bar{x}_n\| \geqq \|z_0 - \bar{x}_n\| - \|x_{n+1} - z_0\| \geqq r - \frac{1}{n^2}$$

by (5). Clearly, $x_{n+1} \in D$. Thus, by induction, we have constructed a sequence $\{x_n\}$ satisfying the condition in Lemma 1 with $c = r > 0$. This implies that C has no normal structure and the proof is complete.

PROPOSITION 2. *A convex subset C of a Banach space has normal structure if and only if it has asymptotic normal structure.*

Proof. The sufficiency part follows from Proposition 1. For necessity, suppose that C has normal structure. Suppose on the contrary that C does not have $\aleph_\lambda$-normal structure for some $\lambda \geqq 0$. By Proposition 1, we must have $\lambda \geqq 1$. Then there exists a nonconstant bounded net $\{x_\alpha\}_{\alpha<\aleph_\lambda}$ in C such that $r(x) = r$ for all $x \in \mathrm{Co}\,(x_\alpha)_{\alpha<\aleph_\lambda}$, where r is the asymptotic radius of $\{x_\alpha\}_{\alpha<\aleph_\lambda}$ in $\mathrm{Co}\,(x_\alpha)_{\alpha<\aleph_\lambda}$. Again $r > 0$.

For each $x \in \mathrm{Co}\,(x_\alpha)_{\alpha<\aleph_\lambda}$, we associate with it an ordinal $\beta(x) < \aleph_\lambda$ as follows: Since $r(x) = r$, $\{\sup_{\alpha\geqq\beta} \|x - x_\alpha\|: \beta < \aleph_\lambda\}$ is a decreasing net in R with infimum r so that there exists an increasing sequence $\{\beta_i\}_{i=1}^\infty$ of ordinals less than $\aleph_\lambda$ with $\lim_i \sup_{\alpha\geqq\beta_i} \|x - x_\alpha\| = r$. We then put $\beta(x) = \sup\{\beta_i: i \geqq 1\}$. Since $\lambda \geqq 1$, $\beta(x) < \aleph_\lambda$.

It follows from the definition of $\beta(x)$ that $\sup\{\|x - x_\alpha\|: \alpha \geqq \gamma\} = r$ for every $\gamma \geqq \beta(x)$. This implies that for every $\varepsilon > 0$, for every $\gamma \geqq \beta(x)$, there exists an ordinal $\alpha > \gamma$ such that $r \geqq \|x_\alpha - x\| \geqq r - \varepsilon$.

Let $u_1 = x_0$. Choose $\alpha > \beta(u_1)$ such that $r \geqq \|x_\alpha - u_1\| \geqq r - 1$ and put $u_2 = x_\alpha$. Suppose that $u_1, \cdots, u_n$, $n \geqq 2$, have been chosen such that $u_i \in \{x_\alpha\}_{\alpha<\aleph_\lambda}$, $\|u_i - u_j\| \leqq r$ $(1 \leqq i \leqq n, 1 \leqq j \leqq n)$ and $\|u_n - \bar{u}_{n-1}\| \geqq r - 1/(n-1)^2$, where $\bar{u}_{n-1} = \sum_{i=1}^{n-1} u_i$. Let $p = \max(\beta(u_1), \cdots, \beta(u_n), \beta(\bar{u}_n))$, where $\bar{u}_n = \sum_{i=1}^n u_i$. Choose $u_{n+1} \in \{x_\alpha\}_{p<\alpha<\aleph_\lambda}$ such that $\|u_{n+1} - \bar{u}_n\| \geqq r - 1/n^2$; this is possible since $p \geqq \beta(\bar{u}_n)$. That $\|u_{n+1} - u_i\| \leqq r$ for every $1 \leqq i \leqq n$ is also true since $p \geqq \beta(u_i)$ for each $i = 1, \cdots, n$. Hence, by induction, we have constructed a sequence $\{u_i\}_{i=1}^\infty \subseteqq \{x_\alpha\}_{\alpha<\aleph_\lambda}$ such that the condition in Lemma 1 is satisfied with $c = r$. This is a contradiction to the assumption that C has normal structure by Lemma 1 and the proof is complete.

REMARK. One can also define γ-normal structure for ordinals $\gamma \geqq 1$ and prove that a convex set has normal structure if and only if it has γ-normal structure for all ordinals $\gamma \geqq 1$.

We are now in the position of proving the following theorem:

THEOREM 1. *Let K be a nonempty weakly compact convex subset of a Banach space and assume that K has normal structure. Let $\mathscr{F}$ be an arbitrary family of commuting nonexpansive maps from K into itself. Then $\mathscr{F}$ has a common fixed point.*

Proof. The theorem is true for $|\mathscr{F}| < \aleph_0$ by Belluce-Kirk's theorem [1], so we assume that $\mathscr{F}$ is infinite. We shall first prove the theorem for $|\mathscr{F}| = \aleph_0$ and assume that it is true for $|\mathscr{F}| = \aleph_\alpha$ for every $\alpha < \gamma$, then prove it for the case $|\mathscr{F}| = \aleph_\gamma$. This would complete the proof by transfinite induction.

Suppose that $|\mathscr{F}| = \aleph_0$ and let $\mathscr{F} = \{f_1, f_2, \cdots\}$. Since closed convex subsets of K are weakly compact, we can apply Zorn's lemma to obtain a subset M of K which is minimal with respect to being nonempty, closed, convex and mapped into itself by every member of $\mathscr{F}$. Suppose M is not a singleton. For each n, let x_n be a common fixed point in M of $\{f_1, \cdots, f_n\}$; this is possible by [1]. Since M is minimal, $\{x_i\}_{i=1}^{\infty}$ is not a constant sequence. Let B be the asymptotic center of $\{x_i\}_{i=1}^{\infty}$ in M. By weakly compactness of M, $B \neq \emptyset$. If $B = M$, then $r(x) = r(y)$ for every x, y in M, in particular $r(x) = r(y)$ for every $x, y \in \mathrm{Co}\,(X_i)_{i=1}^{\infty}$. This implies that the asymptotic center of $\{x_i\}_{i=1}^{\infty}$ in $\mathrm{Co}\,(x_i)_{i=1}^{\infty}$ is $\mathrm{Co}\,(x_i)_{i=1}^{\infty}$ itself—a contradiction to normal structure of K by Proposition 1. Hence B is a proper (closed convex nonempty) subset of M. For every $x \in B$, every $f_n \in \mathscr{F}$, we have $\|x_m - f_n(x)\| = \|f_n(x_m) - f_n(x)\| \leq \|x_m - x\|$ for every $m \geq n$ so that $r(f_n(x)) = \overline{\lim}_m \|x_m - f_n(x)\| \leq \overline{\lim}_m \|x_m - x\| = r(x) = r$, and therefore $f_n(x) \in B$. This shows that B is also mapped into itself by every member of $\mathscr{F}$—a contradiction to the minimality of M. Hence M consists of a single point which must be a common fixed point of $\mathscr{F}$.

Suppose that the theorem is true for all such families $\mathscr{F}$ with $|\mathscr{F}| = \aleph_\alpha$ for some $\alpha < \gamma$. Let $\mathscr{F}$ be a family satisfying the condition in the theorem and $|\mathscr{F}| = \aleph_\gamma$. By previous reasoning, K contains a subset M which is minimal with respect to being nonempty, closed, convex and mapped into itself by each member of $\mathscr{F}$. If M is not a singleton, define a net $\{x_\alpha\}_{\alpha<\aleph_\gamma}$ as follows: For each $\alpha < \aleph_\gamma$, either $|\{f_\alpha\}_{\beta\leq\alpha}| = \aleph_{\alpha_0}$ for some $\alpha_0 < \gamma$ or $|\{f_\beta\}_{\beta\leq\alpha}| < \aleph_0$, so that by induction hypothesis or Belluce-Kirk's theorem, $\{f_\beta\}_{\beta\leq\alpha}$ has a common fixed point which we call x_α. An argument parallel to the previous one together with Proposition 2 shows that the asymptotic center of $\{x_\alpha\}_{\alpha<\aleph_\gamma}$ in M is a proper subset of M and is closed, convex, nonempty and mapped into itself by each member of $\mathscr{F}$. This is a contradiction. Hence M consists of a single point which is a common fixed point of $\mathscr{F}$, completing the proof.

The special feature of the above proof leads to the following

THEOREM 2. *Let K be defined as in Theorem 1. Let $\mathscr{F}$ be a family of nonexpansive self-mappings of K such that every finite subfamily of $\mathscr{F}$ has a common fixed point in every $\mathscr{F}$-invariant closed convex nonempty subset of K. Then $\mathscr{F}$ has a common fixed point.*

REMARK. The technique of asymptotic center can also be applied to prove, by induction, Theorem 1 for finite families through the following facts which had been implicitly used in [4] in the case of uniformly convex Banach spaces.

1. If f is a nonexpansive mapping on a bounded convex set C into itself and $x \in C$, then the asymptotic center of $\{x, f(x), f^2(x), \cdots\}$ in C is mapped into itself by f.

2. If $f_1, \cdots, f_n$ are commuting nonexpansive mappings on a

bounded convex set C into itself, and x is a common fixed point of $f_1, \cdots, f_{n-1}$, then the asymptotic center of $\{x, f_n(x), f_n^2(x), \cdots\}$ in C is mapped into itself by each f_i, $1 \leqq i \leqq n$.

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