Boxes and Marbles

Suppose there are $n$ boxes and $r$ marbles. Find the number of ways the marbles can be placed in the boxes in the following cases:

1. (a) boxes are distinguishable, (b) marbles are distinguishable, (c) any number of marbles in each box.

$$n^r$$

2. (a) boxes are distinguishable, (b) marbles are distinguishable, (c) at most one marble in each box.

$$P(n, r) = \binom{n}{r} \times r!$$

3. (a) boxes are distinguishable, (b) marbles are distinguishable, (c) at least one marble in each box.

$$n!S(r, n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n - i)^r$$

where $S(r, n)$ is the Stirling number of the second kind.

4. (a) boxes are distinguishable, (b) marbles are indistinguishable, (c) any number of marbles in each box.

$$\binom{n + r - 1}{r}$$

5. (a) boxes are distinguishable, (b) marbles are indistinguishable, (c) at most one marble in each box.

$$\binom{n}{r}$$

6. (a) boxes are distinguishable, (b) marbles are indistinguishable, (c) at least one marble in each box.

$$\binom{r - 1}{n - 1}$$
7. (a) boxes are indistinguishable, (b) marbles are distinguishable, (c) any number of marbles in each box.

\[ \sum_{i=1}^{n} S(r, i) \]

8. (a) boxes are indistinguishable, (b) marbles are distinguishable, (c) at most one marble in each box.

\[ \begin{cases} 
1 & \text{if } r \leq n \\ 
0 & \text{if } r > n 
\end{cases} \]

9. (a) boxes are indistinguishable, (b) marbles are distinguishable, (c) at least one marble in each box.

\[ S(r, n) = \frac{1}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n - i)^r \]

10. (a) boxes are indistinguishable, (b) marbles are indistinguishable, (c) any number of marbles in each box.

\[ \sum_{i=1}^{n} p(r, i) \]

where \( p(r, i) \) stands for the number of partitions of the number \( r \) into a sum of \( i \) positive integers (not necessarily distinct). No formulae for \( p(r, i) \) are known, I think.

11. (a) boxes are indistinguishable, (b) marbles are indistinguishable, (c) at most one marble in each box.

\[ \begin{cases} 
1 & \text{if } r \leq n \\ 
0 & \text{if } r > n 
\end{cases} \]

12. (a) boxes are indistinguishable, (b) marbles are indistinguishable, (c) at least one marble in each box.

\[ \sum_{i=1}^{n} p(r - n, i) \]

Fill each box with one marble. Then use Case 10 for the remaining \( r - n \) marbles.