Math 302 Hw3 Solutions

1. Inside a square $ABDE$, take a point $C$ so that $CDE$ is an isosceles triangle with angles $15^\circ$ at $D$ and $E$. What kind of triangle is $ABC$?

**Sol.**

Let $F$ be a point in the square such that $\triangle CDE \cong \triangle FBD$.

Then $CD = DF$

$m_{\angle CDF} = 90^\circ - 15^\circ - 15^\circ = 60^\circ$

$\Rightarrow \triangle CDF$ is equilateral. Therefore $CF = FD = FB \Rightarrow \triangle CFB$ is isosceles. $m_{\angle DFB} = 180^\circ - 15^\circ - 15^\circ = 150^\circ$

$m_{\angle CFD} = 60^\circ$

$\therefore m_{\angle CFB} = 360^\circ - 150^\circ - 60^\circ = 150^\circ$

$\therefore m_{\angle CBF} = \frac{1}{2} \left(180^\circ - 150^\circ\right) = 15^\circ$

$\Rightarrow \therefore m_{\angle ABC} = 90^\circ - 15^\circ - 15^\circ = 60^\circ$.

Similarly $m_{\angle BAC} = 60^\circ$

So $\triangle ABC$ is an equilateral triangle.

2. Prove that any triangle having two equal medians is isosceles.

**Sol.** Suppose $B', C'$ are midpoints of $\overline{AC}, \overline{AB}$ respectively.

And $d = BB' = CC'$. Let $D$ be the two medians intersect at $D$. By theorem, $BD = \frac{2}{3}d$, $DB' = \frac{1}{3}d$

$CD = \frac{1}{3}d$, $DC' = \frac{1}{3}d$.

$\therefore \triangle C'DB \cong \triangle B'DC$ (SAS)

3. Let the three medians be $\overline{AA'}, \overline{BB'}, \overline{CC'}$, intersecting at $P$. By thm,

$BP = \frac{2}{3} BB'$, $CP = \frac{2}{3} CC'$, $AP = \frac{2}{3} AA'$.

$B'C' = \frac{1}{2} BC$, $A'C' = \frac{1}{2} AC$, $A'B' = \frac{1}{2} AB$.

By triangle inequality, one has

$BB' \leq B'C' + C'B = \frac{1}{2} BC + \frac{1}{2} AB$

$CC' \leq C'B' + B'C' = \frac{1}{2} AC + \frac{1}{2} BC$

$AA' \leq AB' + B'A' = \frac{1}{2} AC + \frac{1}{2} AB$.

Adding up, we get

$BB' + CC' + AA' \leq AC + AB + BC = P$

Also

$AB \leq AP + PB = \frac{2}{3} AA' + \frac{2}{3} BB'$

$AC \leq AP + PC = \frac{2}{3} AA' + \frac{2}{3} CC'$

$BC \leq BP + PC = \frac{2}{3} BB' + \frac{2}{3} CC'$

Adding up, we get

$AB + BC + AC \leq \frac{4}{3} AA' + \frac{4}{3} BB' + \frac{4}{3} CC'$

$\therefore \frac{3}{4} (AB + BC + AC) \leq AA' + BB' + CC'$

4. Prove that if $P$ is a point on the circumcircle of a triangle, then the feet of perpendiculars from $P$ to the three sides are collinear.

Sol.

Let $D, E, F$ be the feet of perpendiculars from $P$ to $\overline{AB}, \overline{BC}, \overline{AC}$ respectively.

To prove that $DE, EF$ are collinear, we need to show $m\angle BPD = m\angle FEC$.

Since $ABPC$ is cyclic,

$m\angle BPC + m\angle A = 180^\circ$.
Since \( \angle D \) and \( \angle E \) in \( ADPF \) are \( 90^\circ \), \( ADPF \) is cyclic, so
\[
\angle DPF + \angle A = 180^\circ
\]
\[
: \quad \angle BPC = \angle DPF.
\]
But \( \angle BPC = \angle BPF + \angle FPC \) and
\[
\angle DPF = \angle BPF + \angle DPB,
\]
so \( \angle FPC = \angle DPB \).

Now \( BEPP \) cyclic \( \Rightarrow \angle DPB = \angle DEB \) (they subtend same arc \( \overarc{BD} \))
\( EFPC \) cyclic \( \Rightarrow \angle FPC = \angle FEC \) ("" ""
\( \overarc{PC} \)).

\[
: \quad \angle DEB = \angle FEC
\]

[Note: \( EFCD \) is cyclic b/c \( E, F, P, C \) are all equidistant from the midpoint \( \overarc{PC} \).]

5. Prove the converse of Problem 4, i.e., if the feet of perpendiculars from \( P \) to the three sides are collinear, then \( P \) must be on the circumcircle of the triangle.

Sols:
If \( D, E, F \) are collinear, then
\[
\angle BDE = \angle FEC.
\]
Since \( EBDP \) is cyclic,
\[
\angle BDP = \angle BDE.
\]
Since \( EFPC \) is cyclic
\[
\angle FPC = \angle FEC.
\]
So \( \angle BPD = \angle FPC \)
\[
\Rightarrow \angle DPF = \angle BPC
\]
Since \( ADPF \) is cyclic,
\[
\angle DPF + \angle A = 180^\circ
\]
\[
: \quad \angle BPC + \angle A = 180^\circ
\]
\[
\Rightarrow \quad ABPC \text{ is cyclic } \Rightarrow P \text{ is on the circumcircle of } \triangle ABC.