Proof of cross product formula

$$
\begin{aligned}
|\mathbf{u}\times\mathbf{v}|^2 &= (u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2\\
&= u_2^2v_3^2+u_3^2v_2^2-2u_2u_3v_2v_3+u_3^2v_1^2+u_1^2v_3^2-2u_1u_3v_1v_3+u_1^2v_2^2+u_2^2v_1^2-2u_1u_2v_1v_2
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
|\mathbf{u}|^2|\mathbf{v}|^2 &= (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)\\
&= u_2^2v_3^2+u_3^2v_2^2+u_3^2v_1^2+u_1^2v_3^2+u_1^2v_2^2+u_2^2v_1^2+u_1^1v_1^2+u_2^2v_2^2+u_3^2v_3^2
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathbf{u}\cdot\mathbf{v})^2 &= (u_1v_1+u_2v_2+u_3v_3)^2\\
&= u_1^1v_1^2+u_2^2v_2^2+u_3^2v_3^2+2u_2u_3v_2v_3+2u_1u_3v_1v_3+2u_1u_2v_1v_2
\end{aligned}
$$

Therefore by comparing right hand sides of each, we see that

$$|\mathbf{u}\times\mathbf{v}|^2=|\mathbf{u}|^2|\mathbf{v}|^2-(\mathbf{u}\cdot\mathbf{v})^2$$

Since $\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}||\mathbf{v}|\cos\theta$ and $1-\cos^2\theta=\sin^2\theta$, we see that

$$|\mathbf{u}\times\mathbf{v}|^2=|\mathbf{u}|^2|\mathbf{v}|^2\sin^2\theta$$

Taking square root of both sides, we find

$$|\mathbf{u}\times\mathbf{v}|=|\mathbf{u}||\mathbf{v}|\sin\theta$$