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ON THE LARGEST COMMON FIXED POINT OF A COMMUTING FAMILY OF ISOTONE MAPS

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Abstract. We prove that in a complete partially ordered set with a largest point, every commutative family of isotones has the largest common fixed point. This result for a single mapping was used recently by Ok (2004) to study fixed set theory and its applications in economics.

1. Let $(X, \leq)$ be a partially ordered set. A *largest point* of X is an element a of X such that $x \leq a$ for all $x \in X$. Obviously such a point, if exists, is unique and we could have called it the largest point of X.
In this paper, we call X *complete* if every linearly ordered subset of X has a greatest lower bound in X.
A mapping $f : X \to X$ is called an *isotone* if $f(x) \leq f(y)$ for every $x, y \in X$ with $x \leq y$. A family $\mathcal{F}$ of mappings of X into X is called *commutative* if $f \circ g = g \circ f$ for every $f, g \in \mathcal{F}$, where $\circ$ denotes the composition of maps.

The following theorem and its proof can be found in [1] (see section 8.22, p. 187; reverse the order.):

Theorem 1. *Let $(X, \leq)$ be a nonempty complete partially ordered set which has a largest element. Let f be an isotone of X into X. Then f has a largest fixed point.*

In this paper, we generalize the above theorem to common fixed point of a commutative family of isotones.

The following result is known; we give a proof for the sake of completeness. Here we call a subset S of $(X, \leq)$ a directed set if every two, and hence finitely many, elements of S has a *lower* bound in S.

Proposition 1. *Let $(X, \leq)$ be a complete partially ordered set. Then every directed subset of X has a greatest lower bound in X.*

We begin with the following lemma. For any set S, $|S|$ denotes the cardinality of S.

Lemma 1. *Let D be an infinite directed subset of a partially ordered set $(X, \leq)$. Then there exists a family of directed sets $\{D_\alpha : \alpha \in \Lambda\}$, linearly ordered under inclusion, such that $|D_\alpha| < |D|$ and $\bigcup D_\alpha = D$.*

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Proof. Suppose D is countable and $D = \{x_1, x_2, \dots, x_n, \dots\}$. Define $D_1 = \{x_1\}$. Suppose D_k has been defined such that D_k is finite. Let $a_k \in D$ be a lower bound of $D_k \cup \{x_{k+1}\}$, and let $D_{k+1} = D_k \cup \{x_{k+1}, a_k\}$. By induction, D_n is defined for all positive integers n and that each D_n is finite, has a minimum element, and $D_{n+1} \supset D_n \supset \{x_1, \dots, x_n\}$. It follows that the lemma is true if $|D| = \aleph_0$.
Now suppose that D is uncountable. Let κ be the first uncountable ordinal with cardinality $|D|$. Well order D so $D = \{x_\gamma : \gamma < \kappa\}$.
For an arbitrary infinite set $N \subset D$, define $F(N)$ as follows: For each finite subset $\nu \subset N$, let $u(\nu) \in D$ be a lower bound of ν. Let $F_1(N) = N \cup \{u(\nu) : \nu \subset N \text{ finite }\}$. Let $F_2(N) = F_1(F_1(N)), F_3(N) = F_1(F_2(N)), \dots$, and let $F(N) = \bigcup F_i(N)$. Then $|F(N)| = |N|$ and $F(N)$ is a directed subset of D.
Let ω be the first infinite ordinal. Define $D_\omega = F(\{x_\alpha : \alpha < \omega\})$. Suppose D_α have been defined for all ordinals α with $\omega \le \alpha < \gamma < \kappa$. If γ has a predecessor $\gamma - 1$, then define $D_\gamma = F(D_{\gamma-1} \cup \{x_\alpha : \alpha < \gamma\})$. If γ is a limit ordinal, let $D_\gamma = \bigcup_{\omega \le \alpha < \gamma} D_\alpha$. Note that we have $|D_\alpha| \le |\alpha|$ for each $\alpha < \kappa$ (in case of limit ordinal, $|D_\alpha| \le |\alpha|^2 = |\alpha|$).
Since $\{x_\alpha : \alpha < \gamma\} \subset D_\gamma$, D is the union of $D_\alpha, \omega \le \alpha < \kappa$, each of which is a directed set and has cardinality less than $|\kappa| = |D|$. ☐

Now we prove the proposition:

Proof of Proposition 1. Suppose that every linearly ordered subset of $(X, \le)$ has a greatest lower bound. Suppose on the contrary that there exist directed subsets which do not have greatest lower bound. Let D be such a directed set with minimum cardinality. Obviously D is infinite. By the lemma, D is a union of a family of linearly ordered (under inclusion) of directed subsets $D_\alpha, \alpha \in \Lambda$, of D each of which has cardinality less than $|D|$. By the minimality assumption, each D_α has a greatest lower bound x_α in X. Clearly $S = \{x_\alpha : \alpha < \Lambda\}$ is linearly ordered in X. Hence by completeness, S has a greatest lower bound, which is easily seen to be the greatest lower bound of D. ☐

We are now ready to prove the following theorem:

Theorem 2. *Let $(X, \le)$ be a nonempty complete partially ordered set which has a largest element. Let $\mathcal{F}$ be a commutative family of isotone maps of X into X. Then $\mathcal{F}$ has a largest common fixed point.*

Proof. Let l be the largest element of X. Let $\mathcal{S}$ be the (commutative) semigroup generated by $\mathcal{F}$. Let

$$D = \mathcal{S}(l) = \{s(l) : s \in \mathcal{S}\}.$$

D is a directed subset of X, since $s(l) \le l$ and $t(l) \le l$ imply $s \circ t(l) = t \circ s(l) \le s(l), t(l)$ for any $s, t \in \mathcal{S}$. By completeness assumption and Proposition 1, $\mathcal{S}(l)$ has a greatest lower bound, which we denote by $\bigwedge \mathcal{S}(l)$.
Define a transfinite sequence as follows:

$$x_0 = l, x_1 = \bigwedge \mathcal{S}(l).$$

Suppose x_α have been defined for $\alpha < \gamma$ such that $x_\alpha \le x_\beta$ for $\alpha < \beta < \gamma$. If γ has a predecessor $\gamma - 1$, we define $x_\gamma = \bigwedge \mathcal{S}(x_{\gamma-1})$ (the greatest lower bound exists since $\mathcal{S}(x_{\gamma-1})$ is a directed set as in the case for $x_0 = l$). If γ is a limit ordinal, we define $x_\gamma = \bigwedge \{x_\alpha : \alpha < \gamma\}$. By transfinite induction, x_α is defined for all ordinal α

and satisfies $x_\alpha \leq x_\beta$ for $\alpha < \beta$.
Let κ be an ordinal with $|\kappa| > |X|$. Then $x_\alpha, \alpha < \kappa$, cannot be all distinct, so there exists α_1, β with $\alpha_1 < \beta$ such that $x_{\alpha_1} = x_\beta$. This implies that $x_{\alpha_1} = x_{\alpha_1+1}$. It follows from the definition of x_{α_1+1} and the isotonicity of members of $\mathcal{S}$ that

$$s(x_{\alpha_1}) = x_{\alpha_1}$$

for all $s \in \mathcal{S}$ i.e. x_{α_1} is a common fixed point of $\mathcal{S}$ and hence of $\mathcal{F}$. Let x be a common fixed point of $\mathcal{F}$. Then it is also a common fixed point of $\mathcal{S}$. Certainly $x \leq x_0 = l$. Suppose $x \leq x_\alpha$ for all $\alpha < \gamma$. Then $s(x) = x \leq s(x_\alpha)$ for all $\alpha < \gamma$. It follows that $x \leq x_\gamma$ from the definition of x_γ. Hence by transfinite induction $x \leq x_\alpha$ for all α and so x_{α_1} is the largest common fixed point of $\mathcal{F}$. This completes the proof of the theorem. ☐

Let us remark that the following theorem for commutative family of isotones, which is a special case of Theorem 1 in DeMarr [2], was also used in Ok [3]:

Theorem 3. *Let $(X, \leq)$ be a nonempty complete partially ordered set which has a largest element. Let $\mathcal{F}$ be a commutative family of isotone maps of X into X. Then there exists a minimal $x \in X$ with $x = f(x)$ for all $f \in \mathcal{F}$.*

An extension of DeMarr's result was also made by Wong [4]:

Theorem 4. *Let X be a partially ordered set and let F be a non-empty family of commuting isotone mappings of X into itself. If there exists a well-ordered-complete $x_0 \in X$ with $f(x_0) \leq x_0$ for each $f \in F$, then F has a minimal common fixed point.*

Here, an element $x_0 \in X$ is called *well-ordered-complete* if each well-ordered subset with x_0 as the largest element has an infimum.

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