5.7

314hw4

#6 Multiply by $p = e^{\int f(x)dx}$ to get $py'' + fpy' + (pg + \lambda hp)y = 0$

Then $p' = \frac{d}{dx} e^{\int f(x)dx} = e^{\int f(x)dx} f = fp$, so equation is S-L.

#8 $y'' + \lambda y = 0$; $y'(0) = y'(\pi) = 0$; $r(x) \equiv 1$

1) $\lambda = -k^2 < 0$. Then $y = c_1 e^{kx} + c_2 e^{-kx}$; $c_1 k - c_2 k = 0 = c_1 k e^{k\pi} - c_2 e^{-k\pi}$

$\begin{vmatrix} k & -k \\ ke^{k\pi} & -ke^{-k\pi} \end{vmatrix} = k^2(-e^{-k\pi} + e^{k\pi}) \neq 0$ since $k \neq 0$ $\therefore c_1 = c_2 = 0$

2) $\lambda = 0$. Then $y = c_1 + c_2 x$; $c_2 = 0$. Take $c_1 = 1$; $y_0 = 1$, $\lambda_0 = 0$.

3) $\lambda = k^2 > 0$. Then $y = c_1 \cos kx + c_2 \sin kx$; $c_2 k = 0 = -c_1 k \sin k\pi + c_2 k \cos k\pi$

Take $c_1 = 1$, $k\pi = m\pi$; $y_m = \cos mx$, $\lambda_m = m^2$, $m = 1, 2, \ldots$

$(y_0, y_m) = \int_0^\pi \cos mx\,dx = \frac{1}{m} \sin mx \Big|_0^\pi = 0$

$(y_n, y_m) = \int_0^\pi \cos mx \cos nx\,dx = \int_0^\pi \frac{1}{2}[\cos(m+n)x + \cos(m-n)x]\,dx = 0$

#14 $(xy')' + \lambda x^{-1} y = 0$; $y(1) = 0 = y'(e) = 0$; $r(x) = x^{-1}$

Set $x = e^t$. Then $\frac{dy}{dx} = \frac{dy}{dt}\frac{1}{x}$; $\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\frac{1}{x^2} - \frac{dy}{dt}\frac{1}{x^2}$

$e^t\left(\frac{d^2y}{dt^2} e^{-2t} - \frac{dy}{dt} e^{-2t}\right) + e^t \frac{dy}{dt} e^{-2t} + \lambda e^{-t} y = 0$

$\therefore \frac{d^2y}{dt^2} + \lambda y = 0$; $y(0) = 0 = y'(1)$

1) $\lambda = -k^2 < 0$; $y = c_1 e^{kt} + c_2 e^{-kt}$; $c_1 + c_2 = 0 = c_1 k e^k - c_2 k e^{-k}$ $\therefore c_1 = c_2 = 0$

2) $\lambda = 0$. Then $y = c_1 + c_2 t$; $c_1 = 0 = c_2$

3) $\lambda = k^2 > 0$. Then $y = c_1 \cos kt + c_2 \sin kt$; $c_1 = 0 = c_2 k \cos k$

Take $c_2 = 1$, $k = (\frac{\pi}{2} + m\pi)$; $y_m(t) = \sin[(\frac{\pi}{2} + m\pi)t]$, $\lambda_m = (\frac{\pi}{2} + m\pi)^2$, $m = 0, 1, 2, \ldots$

$y_m(x) = \sin[(\frac{\pi}{2} + m\pi) \ln x]$

$(y_n, y_m) = \int_1^e [\sin[(\frac{\pi}{2} + n\pi)\ln x]\sin[(\frac{\pi}{2} + m\pi)\ln x]\, x^{-1}\,dx$

$= \int_0^1 \sin[(\frac{\pi}{2} + n\pi) t]\sin[(\frac{\pi}{2} + m\pi) t]\,dt$

$= \int_0^1 \frac{1}{2}[\cos(n-m)\pi t - \cos(n+m)\pi t]\,dt = 0$

#18 From prob 6, $p(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

$x^2 y'' + 2xy' + \lambda x^2 y = 0$; $y(\pi) = 0 = y(2\pi)$; $r(x) = x^2$

$y = x^{-1}u$, so $y' = x^{-1}u' - x^{-2}u$; $y'' = x^{-1}u'' - 2x^{-2}u' + 2x^{-3}u$

$xu'' - 2u' + 2x^{-1}u + 2u' - 2x^{-1}u + \lambda x u = 0$

$u'' + \lambda u = 0$; $u(\pi) = 0 = u(2\pi)$

1) $\lambda = k^2 < 0$; $u = c_1 e^{kx} + c_2 e^{-kx}$; $c_1 e^{k\pi} + c_2 e^{-k\pi} = 0 = c_1 e^{2\pi k} + c_2 e^{-2\pi k}$

$\begin{vmatrix} e^{\pi k} & e^{-k\pi} \\ e^{2\pi k} & e^{-2\pi k} \end{vmatrix} = e^{-\pi k} - e^{\pi k} \neq 0$, so $c_1 = c_2 = 0$

2) $\lambda = 0$: $u = c_1 + c_2 x$; $c_1 + c_2\pi = c_1 + c_2 2\pi = 0$; $\Delta \neq 0$, so $c_1 = c_2 = 0$

3) $\lambda = k^2 > 0$: $u = c_1 \cos(kx) + c_2 \sin kx$; $c_1 \cos k\pi + c_2 \sin k\pi = 0 = c_1 \cos k2\pi + c_2 \sin k2\pi$

$\Delta = \cos k\pi \sin k2\pi - \sin k\pi \cos k2\pi = \sin k(2\pi - \pi) = \sin k\pi = 0$ iff $k = m$

So $\lambda_m = m^2$, $u_m = \sin(mx)$, $y_m = \sin(mx)/x$, for $m = 1, 2, \ldots$

$(y_n, y_m) = \int_\pi^{2\pi} x^2 \sin(nx)/x \sin(mx)/x \, dx =$

$= \frac{1}{2}\int_\pi^{2\pi} [\cos(m-n)x - \cos(m+n)x]\, dx = 0$