5.3 314 HW 2

#3 $P_6(x) = \sum_{m=0}^{3} (-1)^m \frac{(12-2m)!\, x^{6-2m}}{2^6 m!(6-m)!(6-2m)!} = \frac{12!\,x^6}{2^6 6!6!} - \frac{10!\,x^4}{2^6 5!4!} + \frac{8!\,x^2}{2^6 2!4!2!} - \frac{6!}{2^6 3!3!}$

$= \frac{231}{16}x^6 - \frac{315}{16}x^4 + \frac{105}{16}x^2 - \frac{5}{16}$

$P_7(x) = \sum_{m=0}^{3} (-1)^m \frac{(14-2m)!\, x^{7-2m}}{2^7 m!(7-m)!(7-2m)!} = \frac{14!\,x^7}{2^7 7!7!} - \frac{12!\,x^5}{2^7 6!5!} + \frac{10!\,x^3}{2^7 2!5!3!} - \frac{8!\,x}{2^7 3!4!}$

$= \frac{429}{16}x^7 - \frac{693}{16}x^5 + \frac{315}{16}x^3 - \frac{35}{16}x$

#4 Assume $s \neq n$, for all s

Then $\left|\frac{a_{s+2}x^{s+2}}{a_s x^s}\right| = \left|\frac{(s-n)(n+s+1)}{(s+2)(s+1)}\right| |x|^2 = \frac{|1-\frac{s}{n}|(\frac{n}{s}+1+\frac{1}{s})|x|^2}{(1+\frac{2}{s})(1+\frac{1}{s})} \to |x|^2$

So converges iff $|x| < 1 = R$

#6 $y_2(x) = P_1(x) = \sum_{m=0}^{0} \frac{(-1)^m(2-2m)!\,x^{n-2m}}{2\,m!(1-m)!(1-2m)!} = \frac{2!\,x}{2} = x$

$a_{s+2} = \frac{(s-1)(s+2)}{(s+2)(s+1)} a_s$ so $a_2 = -a_0$; $a_4 = \frac{a_2}{3} = \frac{-a_0}{3}$; $a_6 = \frac{3a_4}{5} = \frac{-a_0}{5}$

For m even, $a_m = \frac{-a_0}{m-1}$ so $y_1(x) = 1 - \sum_{k=1}^{+\infty} \frac{x^{2k}}{(2k-1)}$

$= 1 - \frac{1}{2}\left(\sum_{m=0}^{+\infty} \frac{x^{m+2}}{m+1}(1+(-1)^m)\right) = 1 - \frac{x}{2}\left(\sum_{m=0}^{+\infty}\frac{x^{m+1}}{m+1} + \sum_{m=0}^{+\infty}\frac{(-1)^m x^{m+1}}{m+1}\right)$

$= 1 - \frac{x}{2}\left[-\ln(1-x) + \ln(1+x)\right] = 1 - \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) = -Q_1(x)$

#7 Let $x = az$ Then $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{a}$ and $\frac{d^2y}{dx^2} = \frac{d}{dz}\left(\frac{dy}{dz}\frac{1}{a}\right)\frac{dz}{dx} = \frac{d^2y}{dz^2}\frac{1}{a^2}$

$\therefore (a^2-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1) = (a^2-a^2z^2)\frac{d^2y}{dz^2}\frac{1}{a^2} - 2az\frac{dy}{dz}\frac{1}{a} + n(n+1) = 0$

$\therefore (1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + n(n+1) = 0$

$\therefore y = a_0 y_1(z) + a_1 y_2(z) = a_0 y_1\left(\frac{x}{a}\right) + a_1 y_2\left(\frac{x}{a}\right)$

5.4

#4 $\sum_{m=0}^{+\infty} 2a_m(m+r)(m+r-1)x^{m+r-1} + \sum_{m=0}^{+\infty} 3a_m(m+r)x^{m+r-1}$

$- \sum_{m=0}^{+\infty} 4a_m(m+r)x^{m+r} + \sum_{m=0}^{+\infty} 2a_m x^{m+r+1} - \sum_{m=0}^{+\infty} 3a_m x^{m+r} = 0$

$a_{s+1}(s+1+r)(2s+2r+3) + 2a_{s-1} - a_s(4s+4r+3) = 0$ for $s \geq -1$

For $s=-1$; $a_0(r)(2r+1) = 0 \quad \therefore r = 0, -\frac{1}{2}$

(over)

For $r=0$, $a_{s+1}(s+1)(2s+3)+2a_{s-1}-a_s(4s+3)=0$ for $s\geq 0$

$a_1=a_0$; $a_2=(-2a_0+7a_1)/10=a_0/2$; $a_3=(-2a_1+11a_2)/21=a_0/6$

$a_4=(-2a_2+15a_3)/36=a_0/24$ In general $a_m=1/m!$

So $y_1=\sum_{m=0}^{+\infty}\frac{x^m}{m!}=e^x$

For $r=-\frac{1}{2}$, $A_{s+1}(s+1)(2s+1)+2A_{s-1}-A_s(4s+1)$ for $s\geq 0$

$A_1=A_0$; $A_2=(-2A_0+5A_1)/6=A_0/2$; $A_3=(-2A_1+9A_2)/15=A_0/6$

$A_4=(-2A_2+13A_3)/28=A_0/24$ In general $A_m=\frac{1}{m!}$

So $y_2=\sum_{m=0}^{+\infty}\frac{x^{m-\frac{1}{2}}}{m!}=e^x/\sqrt{x}$

#8 $\sum_{m=0}^{+\infty}a_m(m+r)(m+r-1)x^{m+r-1}-\sum_{m=0}^{+\infty}a_m x^{m+r}=0$

$a_{s+1}(s+1+r)(s+r)=a_s$ for $s\geq -1$

For $s=-1$: $a_0(r)(r-1)=0$ $\therefore r=0,1$

For $r=1$: $a_{s+1}=a_s/[(s+2)(s+1)]$ for $s\geq 0$

$a_1=a_0/2$; $a_2=a_1/(3\cdot 2)=a_0/(3!\cdot 2)$; $a_3=a_2/(4\cdot 3)=a_0/(4!\,3!)$

In general $a_m=a_0/((m+1)!\,m!)$ $y_1=\sum_{m=0}^{+\infty}\frac{x^{m+1}}{(m!)(m+1)!}$

For $r=0$; $a_{s+1}(s+1)s=a_s$ for $s\geq 0$

$a_1\cdot 0=a_0$ So $a_0=0$, unlucky case

Method 1:

$y_2=u\,y_1$; $x(u''y_1+2u'y_1')=0$; $\frac{u''}{u'}+\frac{2y_1'}{y_1}=0$

$\ln u'+2\ln y_1=c_1$; $u'=1/y_1^2$

$y_1^2=\left[x\left(1+\frac{x}{2}+\frac{x^2}{12}+\frac{x^3}{144}+\cdots\right)\right]^2=x^2\left[1+x+\frac{5x^2}{12}+\frac{7}{72}x^3+\cdots\right]$

$u'=1/y_1^2=\frac{1}{x^2}-\frac{1}{x}+\frac{7}{12}-\frac{19}{72}x\cdots$ $\therefore u=-\frac{1}{x}-\ln x+\frac{7}{12}x-\frac{19}{144}x^2+\cdots$

$\therefore y_2=-\ln x\,y_1+\left(x+\frac{x^2}{2}+\frac{x^3}{12}+\frac{x^4}{144}+\cdots\right)\left(-\frac{1}{x}+\frac{7}{12}x-\frac{19}{144}x^2+\cdots\right)$

$=-\ln x\,y_1-1-\frac{1}{2}x+\frac{1}{2}x^2+\frac{11}{72}x^3+\cdots$

Method 2

$y_3=y_1\ln x+\sum_{m=0}^{+\infty}A_m x^m$; $x\left[-y_1/x^2+2y_1'/x+\sum_{m=0}^{+\infty}A_m m(m-1)x^{m-2}\right]=\sum_{m=0}^{+\infty}A_m x^m$

5.4 # 8 cont. $-\sum_{m=0}^{+\infty} \frac{x^m}{m!(m+1)!} + 2\sum_{m=0}^{+\infty} \frac{x^m}{(m!)^2} + \sum_{m=0}^{+\infty} A_{m+1}(m+1)m x^m = \sum_{m=0}^{+\infty} A_m x^m$

$m = 0: \ -1 + 2 = A_0 = 1$; $m = 1: -\frac{1}{2} + 2 + 2A_2 = A_1$ Let $A_1 = \frac{1}{2}$ $\therefore A_2 = -\frac{1}{2}$

$m = 2: \ -\frac{1}{12} + \frac{1}{2} + 6A_3 = A_2 \therefore A_3 = \frac{-11}{72}$

Thus $y_3 = y_1 \ln x + 1 - \frac{1}{2}x^2 - \frac{11}{72}x^3 + \cdots = -y_2$

#9 $\sum_{m=0}^{+\infty} a_m(m+r)(m+r-1)x^{m+r-1} + \sum_{m=0}^{+\infty} 2a_m(m+r)x^{m+r} + \sum_{m=0}^{+\infty} a_m(m+r)x^{m+r-1}$

$+ \sum_{m=0}^{+\infty} a_m x^{m+r+1} + \sum_{m=0}^{+\infty} a_m x^{m+r} = 0$

$a_{m+1}(m+1+r)^2 + a_m(2m+2r+1) + a_{m-1} = 0$ for $m \geq -1$

Take $m=-1$. Then $r = 0, 0$. So $a_{m+1} = -[a_m(2m+1) + a_{m-1}]/(m+1)^2$ for $m \geq 0$

$a_1 = -a_0$; $a_2 = -[3a_1 + a_0]/4 = a_0/2$; $a_3 = -[5a_2 + a_1]/9 = -a_0/6$

$a_4 = -[7a_3 + a_2]/16 = a_0/4!$ $\qquad a_m = (-1)^m/m!$

$y_1 = \sum_{m=0}^{+\infty} \frac{(-1)^m x^m}{m!} = e^{-x}$

Method 1: $y_2 = u y_1 \therefore x[u'' + 2y_1' u'] + (2x+1)u' y_1 = 0$

$u''/u' + 2y_1'/y_1 + 2 + \frac{1}{x} = 0 \therefore \ln u' + 2\ln y_1 + 2x + \ln x = 0$

$u' = 1/(y_1^2 e^{2x} x)$ Since $y_1^2 e^{2x} = 1$, $u' = \frac{1}{x}$, $u = \ln x$ and $y_2 = \ln x\, e^{-x}$

Method 2: $y_3 = \ln x\, y_1 + \sum_{m=1}^{+\infty} A_m x^m$, $A_0 = 0$

$x[2y_1'/x - y_1/x^2 + \sum_{m=1} A_m m(m-1)x^{m-2}] + (2x+1)(y_1/x + \sum_{m=1}^{+\infty} A_m m x^{m-1}) +$

$(x+1)\sum_{m=1}^{+\infty} A_m x^m = 0$

$2y_1' + 2y_1 + \sum_{m=1}^{+\infty} A_m \cancel{m(m-1)} x^{m-1} + 2\sum_{m=1}^{+\infty} A_m m x^m + \sum_{m=1}^{+\infty} A_m m x^{m-1} + \sum_{m=1}^{+\infty} A_m(x^{m+1} + x^m) = 0$

$y_1' + y_1 = \sum_{m=0}^{+\infty} \frac{(-1)^m m x^{m-1}}{m!} + \sum_{m=0}^{+\infty} \frac{(-1)^m x^m}{m!} = \sum_{\ell=0}^{+\infty} \frac{(-1)^{\ell+1} x^\ell}{\ell!} + \sum_{m=0}^{+\infty} \frac{(-1)^m x^m}{m!} = 0$

$\therefore A_{m+1}(m+1)^2 + A_m(2m+1) + A_{m-1} = 0$ for $m \geq 0$

$\therefore A_1 = 0$; $4A_2 + 3A_1 + A_0 = 0$ so $A_2 = 0$

$A_{m+1} = 0$ since $A_m = A_{m-1} = 0$ Hence $y_3 = \ln x\, y_1 = y_2$

#20 $\sum_{m=0}^{+\infty} 2a_m(m+r)(m+r-1)x^{m+r-1} - \sum_{m=0}^{+\infty} 2a_m(m+r)(m+r-1)x^{m+r}$

$- \sum_{m=0}^{+\infty} a_m(m+r)x^{m+r-1} - \sum_{m=0}^{+\infty} 6a_m(m+r)x^{m+r} - \sum_{m=0}^{+\infty} 2a_m x^{m+r} = 0$

$-2\sum_{m=0}^{+\infty} a_m[(m+r)(m+r+2)+1]x^{m+r} + \sum_{m=0}^{+\infty} a_m(m+r)(2m+2r-3) = 0$

$\therefore 2a_m[(m+r)(m+r+2)+1] = a_{m+1}(m+1+r)(2m+2r-1)$ for $m \geq -1$

for $m=-1$: $a_0(r)(2r-3)=0$, so $r=0, \frac{3}{2}$

For $r=0$: $2a_m(m+1)^2 = a_{m+1}(m+1)(2m-1)$ for $m \geq 0$

$-2a_0 = a_1$; $2a_1 \cdot 4 = a_2 \cdot 2$, so $a_2 = -8a_0$

$2a_2 \cdot 9 = a_3 \cdot 3 \cdot 3$, so $a_3 = -16a_0$

Let $a_0 = 1$, so $y_1 = 1 - 2x - 8x^2 - 16x^3 \ldots$

For $r = \frac{3}{2}$: $2A_m(m^2 + 5m + \frac{25}{4}) = A_{m+1}(m+\frac{5}{2})(m+1)2$

$\frac{25}{4}A_0 = \frac{5}{2}A_1$ so $A_1 = \frac{5}{2}A_0$; $\frac{49}{4}A_1 = 7A_2$, so $A_2 = \frac{35}{8}A_0$

Let $A_0 = 1$, so $y_2 = x^{3/2}(1 + \frac{5}{2}x + \frac{35}{8}x^2 + \ldots)$

This is another example of a hypergeometric equation, with $a=1$, $b=1$, $c=-1/2$.

We did the case $a=b=c=1$ in class

$y_1 = F(1,1,-\frac{1}{2},x)$ as described (16) from problem 18

and y_2 is as part (d) from that problem.