12.8

#4 $f(x) = x$, $B_{mn} = 4\int_0^1\int_0^1 x \sin m\pi x \sin n\pi y \, dx\, dy =$

$4\left[-x\cos(m\pi x)/(m\pi) + \sin(m\pi x)/(m\pi)^2\right]_0^1 \left[-\cos(n\pi y)/(n\pi)^2\right]_0^1$

$4\left[(-1)^{m+1}/(m\pi)\right]\left[1-(-1)^n\right] = \begin{cases} (-1)^{m+1} 8/(mn\pi^2) & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$

So $f(x,y) = \sum_{m=1}^{+\infty}\sum_{k=0}^{+\infty} (-1)^{m+1} 8 \sin(m\pi x)\sin((2k+1)\pi y)/(m(2k+1)\pi^2)$

$= \frac{8}{\pi^2}\Big[\sin \pi x \sin \pi y + \frac{1}{3}\sin\pi x \sin 3\pi y - \frac{1}{2}\sin 2\pi x \sin \pi y$

$- \frac{1}{6}\sin 2\pi x \sin 3\pi y + \frac{1}{3}\sin 3\pi x \sin \pi y + \frac{1}{9}\sin 3\pi x \sin 3\pi y + \cdots\Big]$

#8 $\int_0^1 t(1-t)\sin k\pi t = 2\left[1 - \frac{(-1)^k}{(k\pi)^3}\right] = \frac{4}{(k\pi)^3}$ for k odd and 0 for k even

$\therefore B_{mn} = 4\int_0^1 x(1-x)\sin m\pi x\, dx \int_0^1 y(1-y)\sin n\pi y\, dy$

$= 16/(m^3 n^3 \pi^6)$ for m & n both odd

So $f(x,y) = \frac{16}{\pi^3}\sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty} \sin(2j+1)\pi x \sin(2k+1)\pi x/(2j+1)^3(2k+1)^3$

#18 freq of u_{11} is $\lambda_{11}/2\pi = \frac{c}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}} = \sqrt{\frac{1}{a^2}+\frac{a^2}{A^2}}$

Let $g(a) = \frac{1}{a^2}+\frac{a^2}{A^2}$. Then $g'(a) = \frac{-2}{a^3}+\frac{2a}{A^2} = 0 \Rightarrow A^2 = a^4 \Leftrightarrow \sqrt{A} = a = b$

and $g''(a) = \frac{6}{a^4}+\frac{2}{A^2} > 0$ so crit point is a min.

12.9

#24 $F\ddot{G} = c^2(F_{rr} + \frac{1}{r}F_r + \frac{1}{r^2}F_{\theta\theta})G$ so $\frac{\ddot{G}}{Gc^2} = (F_{rr}+\frac{1}{r}F_r+\frac{1}{r^2}F_{\theta\theta})/F = -k^2$

$\therefore \ddot{G} + (ck)^2 G = 0 = F_{rr} + \frac{1}{r}F_r + \frac{1}{r^2}F_{\theta\theta} + k^2 F = 0$; $\lambda = ck$

$W''Q + \frac{1}{r}W'Q + \frac{1}{r^2}WQ'' + k^2 WQ = 0$ so $(r^2W'' + rW' + k^2r^2)/W = \frac{-Q''}{Q} \equiv m^2$

$\therefore Q'' + m^2 Q = 0 = r^2W'' + rW' + (k^2r^2 - m^2)W = 0$

#26 $u_m(r,\theta,t) = \cos m\theta\, J_m(kr)$ or $\sin m\theta\, J_m(kr)$

so $u_m(R,\theta,t) = 0 \Rightarrow J_m(kR) = 0$ so $k = R/\alpha$ where $J_m(\alpha) = 0$

#28 $\frac{\partial u_{mn}}{\partial t}(r,\theta,0) = (-A_{mn}\, c\, k_{mn}\sin(ck_{mn}0) + B_{mn}\, c\, k_{mn}\cos(ck_{mn}0))$

$\cdot J_m(k_{mn}r)\sin m\theta = B_{mn}\, c\, k_{mn} J_m(k_{mn}r)\sin m\theta = 0 \Rightarrow B_{mn} = 0$

Similarly $B^*_{m,n} = 0$

#29 $\sin n\theta = 0$ so $u^*_{m0} = 0$; $\cos n\theta = 1$, so $u_{m0} = u_m$ in (16)

12.9

#4 $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=\frac{1}{r}\left(\frac{\partial u}{\partial r}+r\frac{\partial^2 u}{\partial r^2}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=\nabla^2 u$ by (5)

#8 $\sin^3\theta=\sin\theta(1-\cos 2\theta)/2=(\sin\theta-(\sin 3\theta-\sin\theta)/2)/2$

$=\frac{3}{4}\sin\theta-\frac{1}{4}\sin 3\theta$; $a_n=0$ since f is odd

$b_n=\frac{800}{\pi}\int_{-\pi}^{\pi}\left[\frac{3}{4}\sin\theta-\frac{1}{4}\sin 3\theta\right]\sin n\theta\,d\theta=\begin{cases}600 & \text{for } n=1\\ -200 & \text{for } n=3\\ 0 & \text{otherwise}\end{cases}$

so $u(r,\theta)=600\,r\sin\theta-200\,r^3\sin 3\theta$

#10 $a_n=0$; $b_n=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\theta\sin n\theta\,d\theta=\frac{1}{\pi}\left[-\frac{\theta\cos n\theta}{n}+\frac{\sin n\theta}{n^2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$=-\cos(n\pi/2)/n+2\sin(n\pi/2)/(\pi n^2)$

first is 0 for n odd and second is 0 for n odd

So $\sum_{k=1}^{+\infty}-\frac{\cos(k\pi)}{(2k)}r^{2k}\sin(2k\theta)+\sum_{k=0}^{+\infty}\frac{2\sin(2k+1)\pi/2}{\pi(2k+1)^2}r^{2k+1}\sin(2k+1)\theta$