12.5 

#10 special case: $U_1 = U_2 = 0$

Then $u(x,t) = \sum_{n=1}^{+\infty} B_n \sin p_n x \, e^{-\lambda_n^2 t} \to 0$ as $t \to +\infty$

Case: $U_1 \neq 0$ or $U_2 \neq 0$

If $k=0$, then $u(x,t) = (Ax+B)(Ct+D)$

$U_1 = B(Ct+D)$; $U_2 = (AL+B)(Ct+D)$ so $BC = 0 = AC$; $BD = U_1$; $AD = (U_2-U_1)/L$

So $u(x,t) = U_1 + (U_2 - U_1)x/L$ is time independent

If $k = -p^2$, then $u(x,t) = (\cos px + B \sin px) e^{-\lambda^2 t}$

So $U_1 = Ae^{-\lambda^2 t}$; $U_2 = (A\cos pL + B\sin pL)e^{-\lambda^2 t}$ is impossible

If $k = p^2$, then $u(x,t) = (Ae^{px} + Be^{-px}) e^{\lambda^2 t}$

So $U_1 = (A+B)e^{\lambda^2 t}$, $U_2 = (Ae^{pL} + Be^{-pL})e^{\lambda^2 t}$ is impossible

I conclude only solution of form $F(x)G(t)$ is one found for $k=0$.

Let $u_1(x,t)$ be that solution and $u_2(x,t)$ be sol where

$u_2(0,t) = 0 = u_2(L,t) = 0$; $u_2(x,0) = f(x) - u_1(x,0)$

Then $u(x,t) = u_1(x,t) + u_2(x,t)$ and $u_2(x,t) \to 0$ as $t \to +\infty$,

by the special case. Hence $u(x,t) \to u_1(x,t) = \boxed{U_1 + (U_2 - U_1)x/L}$

Note that for $u_2(x,t)$, $B_n = \frac{2}{L}\int_0^L (f(x) - u_1(x)) \sin\left(\frac{n\pi x}{L}\right) dx$

#14 $A_0 = \frac{1}{\pi}\int_0^\pi x\,dx = \frac{\pi}{2}$; $A_n = \frac{2}{\pi}\int_0^\pi x \cos nx \, dx =$

$\frac{2}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^\pi = \frac{2}{\pi}\left[(-1)^n - 1\right]/n^2 = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{-4}{n^2\pi} & \text{for } n \text{ odd} \end{cases}$

So $u = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{+\infty} \frac{1}{(2k+1)^2} \cos(2k+1)x \; e^{-(2k+1)^2 t}$

#16 $\frac{2}{\pi}\int_0^\pi \cos 4x \cos nx \, dx = 1$ if $4 = n$ and $0$ otherwise

so $u = 0.5 \cos 4x \, e^{-16t}$

12.6 #2 $f$ is even, so $B(p) = 0$; $A(p) = \frac{2}{\pi}\int_0^{+\infty} e^{-kx} \cos px \, dx =$

$\frac{2 e^{-kx}}{\pi(k^2+p^2)}(-k\cos px + p \sin px)\Big|_0^{+\infty} = \frac{2k}{\pi(p^2+k^2)}$ so $u = \frac{2k}{\pi}\int_0^{+\infty} \frac{\cos px}{p^2+k^2} dp$

#4 $B = 0$; $A = \frac{2}{\pi}\int_0^{+\infty} \frac{\sin x}{x} \cos px \, dx = 1$ if $0 \le p < 1$ and $= 0$ if $1 < p$

So $u(x,t) = \int_0^1 \cos px \, e^{-c^2 p^2 t} dp$

#8 $u(x,t) = \int_0^\pi \cos px \, e^{-c^2 p^2 t} dp$. So $u(x,0) = \int_0^\pi \cos px \, dp = \frac{\sin px}{x}\Big|_0^\pi = \frac{\sin \pi x}{x}$