2.3

#2 $f(x) = k(\sin\pi x - \frac{1}{3}\sin 3\pi x)$, so $A_1 = k$; $A_3 = -\frac{k}{3}$; $A_n = 0$ otherwise

$g(x) = 0$, so $B_n = 0$ so $\boxed{k\cos\pi t \sin\pi x - \frac{k}{3}\cos 3\pi t \sin 3\pi x}$

#6 $A_n/2 = \left[\int_0^{\frac{1}{4}} x \sin n\pi x\,dx + \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{1}{4}\sin n\pi x\,dx + \int_{\frac{3}{4}}^{1}(1-x)\sin n\pi x\,dx\right]$

$= \left[\frac{\sin n\pi x}{(n\pi)^2} - \frac{x\cos n\pi x}{n\pi}\right]_0^{\frac{1}{4}} - \left[\frac{\cos n\pi x}{4n\pi}\right]_{\frac{1}{4}}^{\frac{3}{4}} + \left[-\frac{\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{(n\pi)^2} + \frac{x\cos n\pi x}{n\pi}\right]_{\frac{3}{4}}^{1}$

$= \sin(n\pi/4)/(n\pi)^2 - \cos(n\pi/4)/(4n\pi) - \cos(3n\pi/4)/(4n\pi) + \cos(n\pi/4)/(4n\pi)$

$- \cos(n\pi)/n\pi + \cos(n\pi)/n\pi + \cos(3n\pi/4)/(n\pi) - \frac{3}{4}\cos(3n\pi/4)/(n\pi)$

$+ \sin(3n\pi/4)/(n\pi)^2 = [\sin(n\pi/4) + \sin(3n\pi/4)]/(n\pi)^2 = \begin{cases} 0 & \text{for } n \text{ even} \\ 2\sin(\frac{n\pi}{4})/(n\pi)^2 & \text{for odd} \end{cases}$

So $\sum_{k=0}^{+\infty} \frac{4}{\pi^2}\sin\frac{(2k+1)\pi}{4}\frac{1}{(2k+1)^2}\cos(2k+1)\pi t \sin(2k+1)\pi x =$

$\frac{\sqrt{8}}{\pi^2}\left[\cos t\sin x + \frac{1}{9}\cos 3t \sin 3x - \frac{1}{25}\cos 5t\sin 5x - \frac{1}{49}\cos 7t\sin 7x + \cdots\right]$

#12 $A_n = 0$; $B_n = \frac{.02}{n\pi}\left[\int_0^{\frac{\pi}{2}} x\sin nx\,dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\sin nx\,dx\right]$

$= \frac{.02}{n\pi}\left[\frac{\sin nx}{n^2} - \frac{x\cos nx}{n}\Big|_0^{\frac{\pi}{2}} - \frac{\pi\cos nx}{n} + \frac{x\cos nx}{n} - \frac{\sin nx}{n^2}\Big|_{\frac{\pi}{2}}^{\pi}\right]$

$= \frac{.02}{n\pi}\left[\frac{\sin(n\pi/2)}{n^2} - \frac{\pi\cos(n\pi/2)}{2n} - \pi\cos(n\pi)/n + \pi\cos(n\pi)/n + \pi\cos(n\pi/2)/n\right.$

$\left. - \frac{\pi}{2}\cos(\frac{n\pi}{2})/n^2 + \sin(n\pi/2)/n^2\right] = \frac{.04}{\pi n^3}\sin(n\pi/2)$

$\sum_{n=1}^{+\infty}\frac{.04}{\pi n^3}\sin(n\pi/2)\sin nt \sin nx = \frac{.04}{\pi}\left[\sin t\sin x - \frac{1}{3^3}\sin 3t\sin 3x + \cdots\right]$

#16 $u = F(x)G(t)$ where $F(x) = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x$

$G(t) = a\cos c\beta^2 t + b\sin c\beta^2 t$ by problem 15

$u_t(x,0) = F(x)\dot{G}(t) = F(x)\,b\,c\beta^2 = 0$ so $b = 0$

$u(0,t) = (A + C)G(t) = 0$; $u_{xx}(0,t) = (-A\beta^2 + C\beta^2)G(t) = 0$

So $A = C = 0$. $u(L,t) = (B\sin\beta L + D\sinh\beta L)G(t)$

$u_{xx}(L,t) = (-B\beta^2\sin\beta L + D\beta^2\sinh\beta L)\,G(t)$

$B\sin\beta L + D\sinh\beta L = 0$ has nontrivial solution for B, D
$-B\sin\beta L + D\sinh\beta L = 0$ iff $(\sin\beta L)\sinh(\beta L) = 0$, but $\sinh(\beta L) \ne$

$D = \sin\beta L = 0$; Let $\beta_n = \frac{n\pi}{L}$, $F_n = B_n\sin\beta_n x$, $G_n = a_n\cos c\beta_n^2 t$

12.4

#4 $L=2;\ m=0.9/9.8;\ T=300;\ \rho=m/L;\ c^2=T/\rho$

$\lambda_n = c\,n\pi/L$; period $=2\pi/\lambda_n$; freq $=\lambda_n/2\pi=(c\,n\pi)/(2\pi L)$

$=\sqrt{\frac{T}{\rho}}\,\frac{n}{2L}=\sqrt{\frac{TL}{m}}\,\frac{n}{2L}=\frac{n}{2}\sqrt{\frac{300\cdot 9.8}{0.9\cdot 2}}=\boxed{20.21\,n}$

#12 $u_{xx}-2u_{xy}+u_{yy}=0$; $(y')^2+2y'+1=0$; $y'=1$; $y=-x+\text{const}$

$v=x$; $w=y+x$; $u_{xx}=u_{vv}+2u_{vw}+u_{ww}$

$u_{xy}=2u_{vw}$; $u_{yy}=u_{ww}$ $\therefore u_{vv}=0$

$u=v\,f_1(w)+f_2(w)=x\,f_1(x+y)+f_2(x+y)$

CHECK: $u_x=f_1(x+y)+x\,f_1'(x+y)+f_2'(x+y)$

$u_{xx}=2f_1'(x+y)+x\,f_1''(x+y)+f_2''(x+y)$

$u_{xy}=f_1'(x+y)+x\,f_1''(x+y)+f_2''(x+y)$

$u_y=x\,f_1'(x+y)+f_2'(x+y)$

$u_{yy}=x\,f_1''(x+y)+f_2''(x+y)$ Plug in and get $0=0$

#14 $u_{xx}+u_{xy}-2u_{yy}=0$ $(y')^2-y'-2=0$; $y'=2,-1$

$v=y+x$; $w=y-2x$; $u_{vw}=0$

$u=f_1(v)+f_2(w)=f_1(y+x)+f_2(y-2x)$

#18 $u_{xx}+2u_{xy}+5u_{yy}=0$ $(y')^2-2y'+5=0$; $y'=1\pm 2i$

$v=y-(1-2i)x$; $w=y+(1+2i)x$; $u_{vw}=0$

$u=f_1(v)+f_2(w)=f_1(y-(1-2i)x)+f_2(y-(1+2i)x)$

Take real or imaginary parts to get real solutions.

NOTE hyperbolic + elliptic cases always reduce to u_{vw} and parabolic to u_{vv}