$u_{tt} = c^2 u_{xx}$ (w.e.)

1. Use separation of variables to find eigenfunctions
   Then write $u(x,t)$ as Fourier series.
   Then show $u(x,t) = f_1(x+ct) + f_2(x-ct)$
2. Make change of variables, $v = x+ct$; $w = x-ct$
   w.e. reduces to $u_{vw} = 0$
   So $u = f_1(v) + f_2(w) = f_1(x+ct) + f_2(x-ct)$

3. $u_{xx} = u_{vv}(v_x)^2 + 2u_{vw}(v_x w_x) + u_{ww}(w_x)^2 + u_v v_{xx} + u_w w_{xx}$
   $u_{yy} = u_{vv}(v_y)^2 + 2u_{vw}(v_y w_y) + u_{ww}(w_y)^2 + u_v v_{yy} + u_w w_{yy}$
   $u_{xy} = u_{vv}(v_x v_y) + u_{vw}(v_x w_y) + u_{vw}(w_y v_x) + u_{ww}(w_x w_y) + \cdots$
   If $\theta(x, y(x)) = \text{const}$, then $\theta_x + \theta_y y' = 0$
   For w.e., want $(v_y)^2 = (c v_x)^2$ and $(w_y)^2 = (c w_x)^2$
   So $y' = \pm \frac{1}{c}$ i.e. $x \pm cy = \text{const}$ so $v = x+cy$, $w = x-cy$
   Note $c^2(y')^2 = 1$ is the characteristic equation of w.e.

4. $A u_{xx} + 2B u_{xy} + C u_{yy} = \ldots$
   $A(y')^2 - 2By' + C = 0$ has sol $\Phi(x,y) = \text{const}$, $\Psi(x,y) = \text{const}$
   Let $v = \Phi(x,y)$ and $w = \Psi(x,y)$, so $v_x + v_y y' = 0 = w_x + w_y y'$
   $A u_{xx} + 2B u_{xy} + C u_{yy} = u_{vv}[A(v_x)^2 + 2B v_x v_y + C(v_y)^2] + \cdots$
   $= u_{vv}(v_y)^2[A(y')^2 - 2By' + C] + \cdots = 0 u_{vv} + 0 u_{ww} + \cdots$

#15 $u_{xx} + 2u_{xy} + u_{yy} = 0$; $(y')^2 - 2y' + 1 = 0$; $y' = 1$; $y = x$
Let $v = x$; $w = y - x$; $u_{xx} = u_{vv} - 2u_{vw} + u_{ww}$;
$u_{xy} = u_{vw} - u_{ww}$; $u_{yy} = u_{ww}$
So $u_{xx} + 2u_{xy} + u_{yy} = u_{vv} = 0$ (This always happens in parab. case)
Hence $u = v f_1(w) + f_2(w) = x f_1(y-x) + f_2(y-x)$

#13 $u_{xx} + 9u_{yy} = 0$; $(y')^2 = -9$; $y' = \pm 3i$; $y = \pm 3ix$
Let $v = y + 3ix$; $w = y - 3ix$
$u_{xx} = -9u_{vv} + 18u_{vw} - 9u_{ww}$; $u_{yy} = u_{vv} + 2u_{vw} + u_{ww}$
$\therefore u_{xx} + 9u_{yy} = 36u_{vw} = 0$ $\therefore u = f_1(y+3ix) + f_2(y-3ix)$
To get real valued solutions take real or imaginary part of the above.