12.3 # 4: $f(x)=kx(1-x^2)$; $g(x)=0$; $L=c=1$; $\lambda_n = n\pi = p_n$ 314 h 5

$$B_n = \frac{2}{n\pi}\int_0^1 \sin n\pi x \, g(x)\,dx = 0$$

$$A_n = 2k\int_0^1 (x-x^3)\sin n\pi x\,dx = 2k\left[x^3\frac{\cos n\pi x}{n\pi}\Big|_0^1 + \int_0^1\left(x\sin n\pi x - 3x^2\frac{\cos n\pi x}{n\pi}\right)dx\right]$$

$$= 2k\left[(-1)^n/n\pi - \frac{3x^2\sin n\pi x}{(n\pi)^2}\Big|_0^1 + \int_0^1\left(x\sin n\pi x + 6x\frac{\sin n\pi x}{(n\pi)^2}\right)dx\right.$$

$$= 2k\left[(-1)^n/n\pi - x\left(1+\frac{6}{(n\pi)^2}\right)\frac{\cos n\pi}{n\pi} + \int_0^1\left(1+\frac{6}{(n\pi)^2}\right)\frac{\cos n\pi x}{n\pi}\,dx\right.$$

$$= 2k\left[(-1)^n/n\pi - \left(1+\frac{6}{(n\pi)^2}\right)(-1)^n/n\pi\right] = \frac{12k(-1)^{n+1}}{(n\pi)^3}$$

$$u(x,t) = \sum_{n=1}^{+\infty}\frac{12k(-1)^{n+1}}{(n\pi)^3}\cos(n\pi t)\sin(n\pi x)$$

$$= \frac{12k}{\pi^3}\left[\cos\pi t\sin\pi x - \frac{1}{8}\cos 2\pi t\sin 2\pi x + \frac{1}{27}\cos 3\pi t\sin 3\pi x - \cdots\right)$$

#17 By 16, $u(x,t) = \sum_{n=1}^{+\infty} A_n\cos(c\,p_n^2 t)\sin(p_n x)$ where $p_n = \frac{\pi n}{L}$

$$\sum_{n=1}^{+\infty} A_n \sin(p_n x) = x(L-x)$$

$$A_n = \frac{2}{L}\int_0^L x(L-x)\sin(p_n x)\,dx = \frac{2}{L}\left[x^2\frac{\cos p_n x}{p_n}\Big|_0^L + \int_0^L\left(Lx\sin p_n x - 2x\frac{\cos p_n x}{p_n}\right)dx\right]$$

$$= \frac{2}{L}\left[L^2(-1)^n/p_n - x\left(\frac{L}{p_n}\cos p_n x + 2\frac{\sin p_n x}{(p_n)^2}\right)\Big|_0^L + \int_0^L\left(\frac{L}{p_n}\cos p_n x + \frac{2\sin p_n x}{p_n^2}\right)dx\right]$$

$$= \frac{2}{L}\left\{L^2(-1)^n/p_n - \frac{L^2}{p_n}(-1)^n + \left(\frac{L}{p_n^2}\sin p_n x - \frac{2}{p_n^3}\cos p_n x\right)\Big|_0^L\right)$$

$$= \frac{2}{L}\left[\frac{2}{p_n^3}\left(1-(-1)^{(n)}\right)\right] = \frac{8L^2}{\pi^3 n}\cdot\begin{cases}2 & \text{for } n \text{ odd}\\ 0 & \text{for } n \text{ even}\end{cases}$$

$$u(x,t) = \frac{8L^2}{\pi^3}\sum_{k=0}^{+\infty}\frac{1}{(2k+1)^3}\cos\left(c\frac{\pi^2(2k+1)^2}{L^2}t\right)\sin\left(\frac{\pi(2k+1)}{L}x\right)$$