√ Bessel Equation

314 ho3

$r=\frac{1}{2},\ a_0=\frac{1}{\Gamma(\frac{3}{2})\sqrt{2}}=\frac{1}{\sqrt{2}\sqrt{\pi}/2}=\sqrt{\frac{2}{\pi}}$

For $r=\frac{1}{2}$: $a_s=-a_{s-2}/[s(s+1)]$ for $s\geq 1$

$\therefore a_s=0$ for s odd and $a_{2k}=(-1)^k a_0/(2k+1)!$

$\therefore J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi}}\sum_{k=0}^{+\infty}\frac{(-1)^k x^{2k+\frac{1}{2}}}{(2k+1)!}=\sqrt{\frac{2}{\pi x}}\sin x$

For $r=-\frac{1}{2}$: $A_s\,s(s-1)=-A_{s-2}$ for $s\geq 1$; A_0 and A_1 arbit

$A_2=-A_0/2$; $A_3=-A_1/3!$; $A_{2k}=(-1)^k A_0/(2k)!$; $A_{2k+1}=(-1)^k A_1/(2k+1)!$

$y_2=A_0\sum_{k=0}^{+\infty}\frac{(-1)^k x^{2k-1/2}}{(2k)!}+A_1\sum_{k=0}^{+\infty}\frac{(-1)^k x^{2k+1/2}}{(2k+1)!}$

If $A_0=0$, then $y_2\approx J_{1/2}$ If $A_0=\frac{1}{\Gamma(\frac{1}{2})2^{-1/2}}=\sqrt{\frac{2}{\pi}}$, $A_1=0$, then $y_2=J_{-\frac{1}{2}}=\sqrt{\frac{2}{\pi x}}\cos x$

$r=1,\ J_1(x)=\sum_{k=0}^{+\infty}\frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}$

For $r=-1$, $A_s\,s(s-2)=-A_{s-2}$ for $s\geq 1$

$A_1=0$; $0A_2=A_0$; $A_3=-A_1/3=0$; $A_4=-A_2/4\cdot 2$; $A_6=\frac{A_2}{(6\cdot 4)(4\cdot 2)}$

$A_{2k}=(-1)^{k-1}A_2/[2^{2k-2}k!(k-1)!]$ Take $A_2=-1/2$, then

$y_2=\sum_{k=1}^{+\infty}\frac{(-1)^k x^{2k-1}}{2^{2k-1}k!(k-1)!}=\sum_{s=0}^{+\infty}\frac{(-1)^{s+1}x^{2s+1}}{2^{2s+1}(s+1)!s!}=-J_1(x)$

Let $y_2=J_1\ln x+\sum_{m=0}^{+\infty}A_m x^{m-1}$

$2xJ_1'+\sum_{m=0}^{+\infty}A_m(m-1)(m-2)x^{m-1}+\sum_{m=0}^{+\infty}A_m(m-1)x^{m-1}+\sum_{m=0}^{+\infty}A_m x^{m+1}-\sum_{m=0}^{+\infty}A_m x^{m-1}=0$

$\sum_{k=0}^{+\infty}\frac{(-1)^k(2k+1)x^{2k+1}}{2^{2k}k!(k+1)!}+\sum_{m=0}^{+\infty}[m(m-2)A_m+A_{m-2}]x^{m-1}=0$

$A_1=0$; $A_0=-1$; For m odd $m(m-2)A_m=-A_{m-2}=0$

$\frac{(-1)^k(2k+1)}{2^{2k}k!(k+1)!}+A_{2k+2}(2k+2)(2k)+A_{2k}=0$ for $k\geq 0$

$\frac{-3}{8}+A_4\,8+A_2=0$ Let $A_2=-\frac{1}{4}$. $A_4=\frac{1}{8}[\frac{3}{8}+\frac{1}{4}]=\frac{1}{2^5}[1+\frac{1}{2}+1]$

$\frac{5}{2^4 2\cdot 3!}+A_6\,6\cdot 4+A_4$ so $A_6=-\frac{1}{2^5}[1+\frac{1}{2}+1+\frac{5}{6}]/(6\cdot 4)$

$=-[1+\frac{1}{2}+\frac{1}{3}+1+\frac{1}{2}]/[2^6\cdot 3!\cdot 2]$; $A_{2m}=\frac{(-1)^m(h_m+h_{m-1})}{2^{2m}m!(m-1)!}$ where $h_m=1+\frac{1}{2}+\dots+\frac{1}{m}$

Y_1 is a linear combination of J_1 and y_2

It is given by (8) on 201.