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#6 $\sum_{m=0}^{+\infty} 4a_m(m+r)(m+r-1)x^{m+r-1} + \sum_{m=0}^{+\infty} 2a_m(m+r)x^{m+r-1} + \sum_{m=0}^{+\infty} a_m x^{m+r} = 0$

$4a_s(s+r)(s+r-1) + 2a_s(s+r) + a_{s-1} = 0$ for $s \geq 0$

For $s=0$: $a_0[r(r-1)+2r]=0$ $\therefore r = 0, \frac{1}{2}$

For $r=0$: $a_s = -a_{s-1}/[2s(2s-1)]$ for $s \geq 1$

$a_1 = -a_0/2$; $a_2 = a_0/4!$; $a_3 = -a_0/6!$; $a_m = (-1)^m a_0/(2m)!$

$y_1 = \sum_{m=0}^{+\infty} \frac{(-1)^m x^m}{(2m)!} = \cos\sqrt{x}$

For $r = \frac{1}{2}$: $a_s = -a_{s-1}/[(2s+1)2s]$ for $s \geq 1$

$a_1 = -a_0/(3\cdot 2)$; $a_2 = a_0/5!$; $a_3 = -a_0/7!$; $a_m = (-1)^m a_0/(2m+1)!$

$y_2 = \sum_{m=0}^{+\infty} \frac{(-1)^m x^{m+\frac{1}{2}}}{(2m+1)!} = \text{sen}(\sqrt{x})$

#10 $\sum_{m=0}^{+\infty} (m+r)(m+r-1)x^{m+r} + \sum_{m=0}^{+\infty} 2(m+r)x^{m+r+2} + \sum_{m=0}^{+\infty} a_m x^{m+r+2} - \sum_{m=0}^{+\infty} 2a_m x^{m+r} = 0$

$a_s(s+r)(s+r-1) + 2a_{s-2}(s+r-2) + a_{s-2} - 2a_s = 0$ for $s \geq 0$

For $s=0$: $a_0(r^2-r-2)=0$ $\therefore r = 2, -1$

For $r=2$: $a_s = -a_{s-2}(1+2s)/[s(s+3)]$ for $s \geq 1$

$a_1 = 0$; $a_2 = -a_0/2$; $a_3 = 0$; $a_4 = 9a_0/56$; $a_6 = -13a_0/336$

$y_1 = x^2 - x^4/2 + 9x^6/56 - 13x^8/336 + \cdots$

For $r=-1$: $a_s s(s-3) = a_{s-2}(5-2s)$ for $s \geq 1$

$a_2 = -a_0/2$; $a_3 \cdot 0 = -a_1$ (a_3 arbitrary); $a_4 = 3a_0/8$

$a_6 = -7a_0/48$ Take $a_0 = 1$; $a_3 = 0$

$y_2 = x^{-1} - x/2 + 3x^3/8 - 7x^5/48 + \cdots$

$$x(1-x)y''+(1-3x)y'-y=0$$

$$\sum_{m=a_0}^{+\infty} a_m(m+r)(m+r-1)x^{m+r-1}-\sum_{m=0}^{+\infty} a_m(m+r)(m+r-1)x^{m+r}$$
$$+\sum_{m=0} a_m(m+r)x^{m+r-1}-\sum_{m=0}^{+\infty}3a_m(m+r)x^{m+r}-\sum_{m=0}^{+\infty}a_m x^{m+r}=0$$

$$a_{s+1}(s+1+r)(s+r)-a_s(s+r)(s+r-1)+a_{s+1}(s+1+r)-3a_s(s+r)-a_s=0 \quad \text{for } s\ge -1$$

For $s=-1$: $a_0[r^2-r+r]=0 \therefore r=0,0$

$$a_{s+1}(s+1)^2=a_s(s+1)^2 \text{ for } s\ge 0 \therefore a_m=a_0$$

$$y_1=\sum_{m=0}^{+\infty}x^m$$

Method 1: $y_2=uy_1$; $y_2'=uy_1'+u'y_1$; $y_2''=uy_1''+2u'y_1'+u''y_1$

$$x(1-x)[2u'y_1'+u''y_1]+(1-3x)u'y_1=0$$

$$2\frac{y_1'}{y_1}+\frac{u''}{u'}=\frac{3x-1}{x(1-x)}=\frac{-1}{x}+\frac{2}{1-x}$$

$$\therefore 2\ln y_1+\ln u'=-\ln x-2\ln(1-x) \therefore u'=1/[x(1-x)^2y_1^2]$$

$$(1-x)^2y_1^2=(1-2x+x^2)\sum_{m=0}^{+\infty}(m+1)x^m=\sum_{m=0}^{+\infty}(m+1)x^m-2\sum_{s=1}^{+\infty}s x^s+\sum_{s=2}^{+\infty}(s-1)x^s$$

$$=1+\sum_{s=1}^{+\infty}[s+1-2s+s-1]=1 \therefore u'=\frac{1}{x};\ u=\ln x+c$$

Take $c=0$, so $y_2=(\ln x)y_1$

Method 2: $y_2=y_1\ln x+\sum_{m=1}^{+\infty}A_m x^m \quad (A_0=0)$

$$y_2'=\ln x\, y_1'+\frac{1}{x}y_1+\sum_{m=1}^{+\infty}A_m m x^{m-1}$$

$$y_2''=\ln x\, y_1''+\frac{2}{x}y_1'-\frac{1}{x^2}y_1+\sum_{m=1}^{+\infty}A_m m(m-1)x^{m-2}$$

$$x(1-x)[2y_1'/x-y_1/x^2]+\sum_{m=1}^{+\infty}A_m m(m-1)x^{m-1}-\sum_{m=1}^{+\infty}A_m m(m-1)x^m$$
$$+(1-3x)y_1/x+\sum_{m=1}^{+\infty}A_m m x^{m-1}-3\sum_{m=1}^{+\infty}A_m m x^m-\sum_{m=1}^{+\infty}A_m x^m=0$$

$$2(1-x)y_1'-2y_1+\sum_{m=1}^{+\infty}A_m m^2x^{m-1}+\sum_{m=1}^{+\infty}-A_m(m+1)^2x^m=0$$

$$2\left[\sum_{m=0}^{+\infty}mx^{m-1}-\sum_{m=0}^{+\infty}mx^m-\sum_{m=0}^{+\infty}x^m\right]+\sum_{m=0}^{+\infty}A_{m+1}(m+1)^2x^m-\sum_{m=0}^{+\infty}A_m(m+1)^2x^m=0$$

$$\therefore A_{m+1}=A_m=0 \text{ for } m\ge 0$$

$\therefore y_2=y_1\ln x$ as above

Note: Most solutions won't turn out this simple

$x^2y'' + x^2y' - \frac{3}{4}y = 0$; 0 is a reg sing pt, so $y = \sum_{k=0}^{\infty} C_k x^{k+r}$

$C_k\left[(k+r)(k+r-1)-\frac{3}{4}\right] + C_{k-1}(k-1+r) = 0$ for $k \geq 0$

With $k = 0$, $r^2 - r - \frac{3}{4} = 0$. so $r = \frac{3}{2}, -\frac{1}{2}$. (Note $\frac{3}{2} - (-\frac{1}{2}) = 2$. possible danger!)

Take $r = -\frac{1}{2}$. $C_k[k^2 - 2k] + C_{k-1}(k-\frac{3}{2}) = 0$ for $k \geq 1$.

Then $-C_1 - \frac{1}{2}C_0 = 0$; $C_2 \cdot 0 + C_1 \frac{1}{2} = 0$. So $C_1 = C_0 = 0$.

Thus we are unlucky. (You knew in your heart that that would happen.)

Continue with C_2 arbitrary or start over with $r = \frac{3}{2}$.

$C_k[k^2 + 2k] + C_{k-1}(k + \frac{1}{2}) = 0$ for $k \geq 1$.

$C_1 = \frac{-3\ C_0}{2 \cdot 3}$; $C_2 = \frac{5 \cdot 3\ C_0}{2 \cdot 2!\,4!}$; $C_3 = \frac{-7 \cdot 5 \cdot 3\ C_0}{2^2 \cdot 3!\,5!}$. Let $C_0 = 1$

Then $\boxed{y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n\ 1 \cdot 3 \cdot 5 \cdots (2n+1)}{2^{n-1}\ n!\,(n+2)!} x^{n+\frac{3}{2}} = x^{\frac{3}{2}} - \frac{1}{2}x^{\frac{5}{2}} + \frac{5}{2^5}x^{\frac{7}{2}} - \frac{7}{2^6 \cdot 3}x^{\frac{9}{2}}}$

Now let $y_2 = vy_1$. Then $y_1v'' + [2y_1' + y_1]v' = 0$

So $v' = e^{-x}/y_1^{\,2}$; $y_1^{\,2} = x^3 - x^4 + \frac{9}{16}x^5 - \frac{11}{48}x^6 + \cdots$

Thus $v' = x^{-3} - \frac{1}{16}x^{-1} - 0 + \cdots$. So $v = -\frac{1}{2}x^{-2} - \frac{1}{16}\ln x + 0x + \cdots$

Hence $\boxed{y_2 = \frac{-1}{16}y_1 \ln x - \frac{1}{2}x^{\frac{-1}{2}} + \frac{1}{4}x^{\frac{1}{2}} - \frac{5}{64}x^{\frac{3}{2}} + \frac{7}{384}x^{\frac{5}{2}} + \cdots}$

Now assume that $y_2 = y_1 \ln x + \sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}$

Then $0 = \sum_{k=0}^{\infty} b_k(k-\frac{1}{2})(k-\frac{3}{2})x^{k-\frac{1}{2}} + 2xy_1' + (x-1)y_1 + \sum_{k=0}^{\infty} b_k(k-\frac{1}{2})x^{k+\frac{1}{2}} - \frac{3}{4}\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}$

So $\sum_{k=1}^{\infty}\left[b_k(k-2)k + b_{k-1}(k-\frac{3}{2})\right]x^{k-\frac{1}{2}} + \sum_{k=1}^{\infty}\left[C_{k-2}2(k-1) + C_{k-3}\right]x^{k-\frac{1}{2}} = 0$

Thus $b_k k(k-2) = -\left[C_{k-2}2(k-1) + C_{k-3} + b_{k-1}(k-\frac{3}{2})\right]$ for $k \geq 1$.

So $-b_1 = \frac{1}{2}b_0$; $0 = 2C_0 + \frac{1}{2}b_1$; So $b_1 = -4$, $b_0 = 8$. Let $b_2 = \frac{5}{4}$.

$3b_3 = -\left[4C_1 + C_0 + \frac{3}{2}b_2\right] = -\left[-2 + 1 + \frac{15}{8}\right] = \frac{-7}{8}$. So $b_3 = \frac{-7}{24}$

Hence $\boxed{y_2 = y_1 \ln x + 8x^{-1/2} - 4x^{1/2} + \frac{5}{4}x^{3/2} - \frac{7}{24}x^{5/2} + \cdots}$

Note that **this y_2 is -16 times** the one found by the first method. Since b_2 was undetermined, it was chosen so they would agree.

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Do problem 10 on page 310 by each of the two methods of section 7.1 and by the method of section 7.4.