Math 314 April 21, 2009

This part is closed book. You may use 1 page of notes. Follow the Honor Code.

(7) 1. Let f(x) = x^2 for 0 < x < 3 and have period 3.

a) Sketch the graph of f for -6 < x < 6.

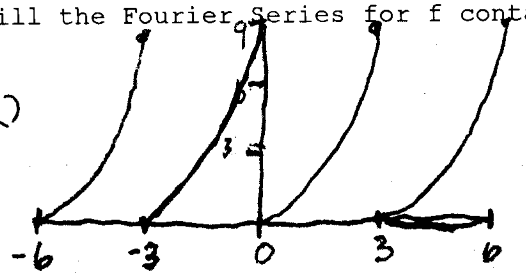
b) Can f be represented by a power series near 0? by a trig series? Explain.

c) If the answer to either question is yes, to what value will it converge at 0?

d) Is f even or odd?

e) Will the Fourier Series for f contain cosine terms? sine terms? Explain.

a)



b) power series: NO
It is not continuous at 0.
trig series: YES
It is piecewise continuous

c) Fourier series converges to $\frac{1}{2}(0+9) = 4.5$

d) $f(-1) = f(2) = 4 \neq \pm f(1) = \pm 1$
So neither

e) Only odd functions contain only sine terms
" even " " " cosine "
So series for f contains both types

(16) 2. Find the Fourier cosine series and the sine series for the periodic extensions of f(x) = 1 for 0 < x < π.

a) $f(x) = 1 = a_0 + \sum_{n=1}^{+\infty} a_n \cos nx$ so $a_0 = 1;\ a_n = 0$ for $n = 1$

So cosine series is $\boxed{1}$

b) $f(x) = \sum_{n=1}^{+\infty} b_n \sin nx \ ; \ b_n = \frac{2}{\pi}\int_0^1 \sin nx\,dx = \frac{-2}{\pi}\frac{\cos nx}{n}\Big|_0^1$

$= \frac{2}{n\pi}\left[1-(-1)^n\right] = \begin{cases} 0 & \text{for } n \text{ even} \\ 4/(\pi n) & \text{for } n \text{ odd} \end{cases}$

So sine series is $\boxed{\sum_{k=0}^{+\infty} \frac{4}{\pi(2k+1)} \sin(2k+1)x}$

(8) 3. y" + 0.01y = r(t) where r(t) = -1 for $-\pi < t < 0$, r(t) = 1 for $0 < t < \pi$, and is 2π periodic. Find the general solution.

solution y_h of $y'' + .01y = 0$ is $y_h = c_1 \cos(.1t) + C_2 \sin(.1t)$

by #2: $r(t) = \sum_{k=0}^{+\infty} \frac{4}{\pi(2k+1)} \sin(2k+1)t$

assume $y_p = \sum_{n=0}^{+\infty} (A_n \cos nt + B_n \sin nt)$

$\therefore y_p'' + 0.01 y_p = \sum_{n=0}^{+\infty} [A_n(.01 - n^2)\cos nt + B_n(.01 - n^2)\sin nt] = r(t)$

$\therefore A_n = 0 \quad B_{2k+1} = \frac{4}{\pi(2k+1)(.01 - (2k+1)^2)}$

$y = c_1 \cos(.1t) + C_2 \sin(.1t) + \sum_{k=0}^{+\infty} \frac{4 \sin(2k+1)t}{\pi(2k+1)(.01 - (2k+1)^2)}$

(8) 4. Let f(x) = 1 for 0 < x < 1 and f(x) = 0 for 1 < x.
a) Find the Fourier cosine integral for f.
b) Use your answer to evaluate $\int_0^{+\infty} \frac{\sin w \cos w}{w} dw$.

a) $A(w) = \frac{2}{\pi} \int_0^1 \cos wx \, dx = \frac{2 \sin wx}{\pi w} \Big|_0^1 = \frac{2 \sin w}{\pi w}$

$\therefore f(x) = \int_0^{+\infty} \frac{2 \sin w \cos wx}{\pi w} dw$

b) Taking $x = 1$: $\frac{1}{2} = f(1) = \int_0^{+\infty} \frac{2 \sin w \cos w}{\pi w} dw$

$\therefore \int_0^{+\infty} \frac{\sin w \cos w}{w} dw = \boxed{\frac{\pi}{4}}$

(5) 5. Find the Fourier transform of f where f(x) = e^(-ax) for 0 < x, f(x) = 0 for 0 > x, and 0 < a.

$\mathcal{F}(f)(w) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-ax} e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-(a+iw)x} dx$

$= \left(\frac{1}{\sqrt{2\pi}}\right) \frac{-e^{-(a+iw)x}}{iw+a} \Big|_0^{+\infty} = \boxed{\frac{1}{\sqrt{2\pi}(iw+a)}}$

since $e^{-(a+iw)x} \to 0$ as $x \to +\infty$ for $0 < a$

(5) 6. Show that u = ln(x^2 + y^2) is harmonic, i.e. $u_{xx} + u_{yy} = 0$.

$$u_x = \frac{2x}{(x^2+y^2)}$$

$$u_{xx} = \frac{2(x^2+y^2) - (2x)(2x)}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

$$u_{yy} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$$

$$\therefore u_{xx} + u_{yy} = 0$$

(6) 7. Find harmonic functions by separation of variables. Remember there are 3 cases.

Assume $u = F(x)G(y)$; $u_{xx} = F''G$; $u_{yy} = FG''$

$\therefore \frac{F''}{F} = \frac{-G''}{G} = k$; $F'' - kF = 0 = G'' + kG$

1) $k = 0$. Then $u(x,y) = (ax+b)(cy+d)$

2) $0 > k = -p^2$. Then $u(x,y) = (a \cos px + b \sin px)(ce^{py} + de^{-py})$

3) $0 < k = p^2$ Then $u(x,y) = (ae^{px} + be^{-px})(c \cos py + d \sin py)$

NOTE: We are <u>not</u> claiming all harmonic functions can be obtained this way. For example, the function in #6 or some of the ones in #8.

(5) 8. Find harmonic functions by reducing the equation to normal form.
You may assume hyperbolic and elliptic reduce to u_{vw} and parabolic reduces to u_{vv}.

$u_{xx} + u_{yy} = 0$; $(y')^2 + 1 = 0$; $y' = \pm i$

$y \pm ix = \text{const}$ Let $v = y + ix$; $w = y - ix$

$u_{xx} = -u_{vv} + 2u_{vw} - u_{ww}$

$u_{yy} = u_{vv} + 2u_{vw} + u_{ww}$

$\therefore u_{xx} + u_{yy} = 4u_{vw} = 0$ (as predicted)

$u = f_1(v) + f_2(w) = f_1(y+ix) + f_2(y-ix)$

Take real or imaginary parts to get harmonic functions

Chapter 11 314 Take Home 2

#12 $f(x)=\begin{cases}0 \text{ if } -\frac{\pi}{2}<x<\frac{\pi}{2}\\ 1 \text{ if } -\pi<x<-\frac{\pi}{2} \text{ or } \frac{\pi}{2}<x<\pi\end{cases}$ f is even so $b_n=0$

(10)

$a_0=\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} 1\,dx=\frac{1}{2}$; $a_n=\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}\cos nx\,dx=\frac{2}{n\pi}(\sin nx)\Big|_{\frac{\pi}{2}}^{\pi}=-\frac{2}{n\pi}\sin\frac{n\pi}{2}$

$\therefore f(x)=\boxed{\frac{1}{2}-\frac{2}{\pi}\sum_{n=1}^{+\infty}(\sin\frac{n\pi}{2})/n\cos nx}=\frac{1}{2}-\frac{2}{\pi}[\cos x-\cos 3x/3+\cos 5x/5 - \cdots]$

#24 $\int_{-\pi}^{\pi} f^2\,dx=\pi$ $\therefore$ $\frac{1}{\pi}\pi=2(\frac{1}{2})^2+\sum_{n=1}^{\infty}a_n^2$ $\therefore$ $\frac{1}{2}=\frac{4}{\pi^2}\sum_{k=0}^{+\infty}\frac{1}{(2k+1)^2}$

(6) Hence $\boxed{\frac{\pi^2}{8}}=\sum_{k=0}^{+\infty}\frac{1}{(2k+1)^2}$

#22 By 17, $|\sin 8\pi x|=\frac{2}{\pi}-\frac{4}{\pi}\sum_{k=1}^{+\infty}\frac{\cos 16k\pi x}{(2k-1)(2k+1)}$ for $-\frac{1}{8}<x<\frac{1}{8}$

(6) Take $x=0$. Then $\sum_{k=1}^{+\infty}\frac{1}{(2k-1)(2k+1)}=\frac{2}{\pi}\frac{\pi}{4}=\boxed{\frac{1}{2}}$.

#30 $r(t)=t(\pi^2-t^2)$ for $(-\pi<t<\pi)$ is odd so $a_n=0$

(10)

$b_n=\frac{2}{\pi}\int_0^{\pi}(\pi^2 t-t^3)\sin nt\,dt=\frac{2}{\pi}[t^3\frac{\cos nt}{n}\Big|_0^{\pi}+\int_0^{\pi}(\pi^2 t\sin nt-3t^2\frac{\cos nt}{n})dt$

$=\frac{2}{\pi}[\pi^3\frac{(-1)^n}{n}-3t^2\frac{\sin nt}{n^2}\Big|_0^{\pi}+\int_0^{\pi}(\pi^2 t+6t/n^2)\sin nt\,dt$

$=\frac{2}{\pi}[\pi^3\frac{(-1)^n}{n}-(\pi^2+6/n^2)t\cos nt/n\Big|_0^{\pi}+\int_0^{\pi}(\pi^2+6/n^2)\cos nt/n\,dt$

$=\frac{2}{\pi}[\pi^3\frac{(-1)^n}{n}-(\pi^2+6/n^2)\frac{\pi}{n}(-1)^n]=\frac{12}{n^3}(-1)^{n+1}$

Assume $y_p=\sum_{n=1}^{+\infty}(A_n\cos nt+B_n\sin nt)$ Then

$y''+w^2y=\sum_{n=1}^{+\infty}[A_n(w^2-n^2)\cos nt+B_n(w^2-n^2)\sin nt]=r(t)$

$=\sum_{n=1}^{+\infty}\frac{12}{n^3}(-1)^{n+1}\sin nt$ $\therefore$ $A_n=0$; $B_n=\frac{12}{n^3}(-1)^{n+1}/(w^2-n^2)$

$y_h=c_1\cos wt+c_2\sin wt$ So $y=c_1\cos wt+c_2\sin wt+$

$\sum_{n=1}^{+\infty}\frac{12}{n^3}(-1)^{n+1}/(w^2-n^2)\sin nt$

#34 $A(w)=\frac{2}{\pi}\int_0^2(1-x/2)\cos wx\,dx=\frac{2}{\pi}[\frac{\sin wx}{w}-\frac{x}{2}\sin wx\Big|_0^2+\frac{1}{2}\int_0^2\frac{\sin wx}{w}dx]$

(6) $=\frac{2}{\pi}[\frac{\sin 2w}{w}-\frac{\sin 2w}{w}-\frac{1}{2}\frac{\cos wx}{w^2}\Big|_0^2]=(1-\cos 2w)/(\pi w^2)$

$f(x)=\int_0^{+\infty}\frac{(1-\cos 2w)}{\pi w^2}\cos wx\,dw$

Chapter 12

#20 $u_{xy} + u_y + x + y + 1 = 0$

(6) Integrate wrt y: $u_x + u + xy + \frac{y^2}{2} + y = C_1(x)$

This is first order linear in x. $u_h = C_2(y) e^{-x}$

Try $u_p = A(y)x + B(y) + C(x)$

Then $A(y) + C'(x) + A(y)x + B(y) + C(x) + xy + \frac{y^2}{2} + y = C_1(x)$

$\therefore A(y) = -y$; $B(y) = -\frac{y^2}{2}$ so $u_p = -yx - \frac{y^2}{2} + C(x)$

$$\boxed{\text{Hence } u = C_2(y) e^{-x} - yx - \frac{y^2}{2} + C(x)}$$

Check: $u_x = -C_2(y) e^{-x} - y + C'(x)$

$u_{xy} = -C_2'(y) e^{-x} - 1$

$u_y = C_2'(y) e^{-x} - x - y$

$u_{xy} + u_y = x + y + 1$

#40 $2u_{xx} + 5u_{xy} + 2u_{yy} = 0$

(6) $2(y')^2 - 5y' + 2 = 0$; $y' = \frac{5 \pm \sqrt{25-16}}{4} = \frac{1}{2}, 2$

$\therefore y = \frac{1}{2}x + \text{const}$; $y = 2x + \text{const}$

$v = y - \frac{1}{2}x$; $w = y - 2x$ Hyperbolic so $u_{vw} = 0$

$u = f_1(v) + f_2(w) = \boxed{f_1(y - x/2) + f_2(y - 2x)}$