Math 314
W. T. Kiley

In Class Exam 1
March 3, 2009

Closed book. One page of notes.

(12)

1. Use series around x = 0 to solve:
a) y" - 4 y' = 0; b) x^2y" + xy' + (x^2 - 9/4)y = 0.

a) $y = \sum_{m=0}^{+\infty} a_m x^m$; $\sum_{m=0}^{+\infty} a_m m(m-1)x^{m-2} = 4\sum_{m=0}^{+\infty} a_m m x^{m-1}$

$a_{m+1}(m+1)m = 4a_m m$ for $m \geq 0$

a_0, a_1 undetermind ; $a_2 = \frac{4a_1}{2}$; $a_m = \frac{4^{m-1}}{m!} a_1$

$$y = a_0 + a_1 \sum_{m=1}^{+\infty} \frac{4^{m-1}x^m}{m!} = a_0 + \frac{a_1}{4}\left[e^{4x} - 1\right]$$

b) $y = \sum_{m=0}^{+\infty} a_m x^{m+r}$ Note: This is Bessel with $\nu = \frac{3}{2}$

$$\sum_{m=0}^{+\infty} a_m (m+r)(m+r-1)x^{m+r} + \sum_{m=0}^{+\infty} a_m (m+r)x^{m+r} + \sum_{m=0}^{+\infty} a_m x^{m+r+2} +$$
$$-\frac{9}{4}\sum_{m=0}^{+\infty} a_m x^{m+r} = 0$$

$a_m\left[(m+r)^2 - \frac{9}{4}\right] = -a_{m-2}$ for $m \geq 0$

$a_0\left[r^2 - \frac{9}{4}\right] = 0$ ∴ $r = \pm\frac{3}{2}$

Take $r = -3/2$; $a_m(m-3)m = -a_{m-2}$

$a_1 = 0$; $a_2 = a_0/2$; a_3 arbitrary (lucky!) ; $a_4 = \frac{-a_2}{4} = \frac{-a_0}{4\cdot 2}$

$a_5 = -a_3/(5\cdot 2)$; $a_6 = -a_4/(6\cdot 3) = a_0/(6\cdot 4\cdot 3\cdot 2)$

$a_7 = -a_5/(7\cdot 4) = a_3/(7\cdot 5\cdot 4\cdot 2)$

$$y = a_0 x^{-3/2}\left(1 + \frac{x^2}{2} - \frac{x^4}{4\cdot 2} + \frac{x^6}{6\cdot 4\cdot 3\cdot 2} + \cdots\right) + a_3 x^{3/2}\left(1 - \frac{x^2}{5\cdot 2} + \frac{x^4}{7\cdot 5\cdot 4\cdot 2} + \cdots\right)$$

$$= a_0 x^{-3/2}\sum_{k=0}^{+\infty}\frac{(-1)^k x^{2k}}{2^{2k}k!\left(-\frac{1}{2}\right)\cdots\left(-\frac{1}{2}+k\right)} + a_3 x^{3/2}\sum_{k=0}^{+\infty}\frac{(-1)^k x^{2k}}{2^{2k}k!\left(\frac{5}{2}\right)\cdots\left(\frac{5}{2}+k\right)}$$

(8)
2. Solve x^2y" + xy' + (x^2 - 1)9y/4 = 0 by letting z =3x/2.

$\frac{dy}{dx} = \frac{dy}{dz}\left(\frac{3}{2}\right)$ $\quad \left(\frac{2z}{3}\right)^2 \frac{d^2y}{dz^2}\frac{9}{4} + \left(\frac{2z}{3}\right)\frac{dy}{dz}\frac{3}{2} + \left(\left(\frac{2z}{3}\right)^2 - 1\right)9y/4 = 0$

$\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\frac{9}{4}$ $\quad z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - 9/4)y = 0$

$\therefore y = C_1 J_{\frac{3}{2}}(z) + C_2 J_{-\frac{3}{2}}(z) = C_1 J_{\frac{3}{2}}\left(\frac{3x}{2}\right) + C_2 J_{-3/2}\left(\frac{3x}{2}\right)$

(10)
3. Let $y_1(x) = x$ and $y_2(x) = (3x^2 - 1)/2$ on [-1, 1] with r(x) = 1.
a) Find $\|y_1\|$; b) Determine whether y_1 and y_2 are orthogonal.

a) $\|y_1\|^2 = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$ so $\|y_1\| = \sqrt{2/3}$

b) $(y_1, y_2) = \int_{-1}^{1} x(3x^2-1)/2\,dx = \frac{1}{2}\left[\frac{3}{4}x^4 - \frac{x^2}{2}\right]_{-1}^{1} = 0$

$\therefore y_1 \perp y_2$ Note: This are Legendre $P_1(x) + P_2(x)$

(10)
4. Find the Fourier coefficients of 1, cos x, and sin x for the representation of x on [-π,π].

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{2\pi}\frac{x^2}{2}\Big|_{-\pi}^{\pi} = 0$

$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x\,dx = 0$ since $x\cos x$ is an odd function

$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\,dx = \frac{1}{\pi}\left[-x\cos x\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\cos x\,dx\right] = \frac{2\pi}{\pi} = 2$

(10)
5. Find the positive eigenvalues and corresponding eigenfunctions for:
y" +λ y = 0; y(0) = y(1); y'(0) = y'(1).

$\lambda = k^2$; $y = C_1\cos kx + C_2\sin kx$

$y' = -kC_1\sin kx + kC_2\cos kx$

$C_1 = C_1\cos k + C_2\sin k$ $\quad \begin{bmatrix}\cos k - 1 & \sin k \\ -\sin k & \cos k - 1\end{bmatrix} = (\cos k - 1)^2 + \sin^2 k$

$kC_2 = -kC_1\sin k + kC_2\cos k$

$= \cos^2 k - 2\cos k + 1 + \sin^2 k = 0$ iff $2\cos k = 2$ iff $k = (2m\pi)$.

So $C_1 = C_1\cdot 1 + C_2\cdot 0$

$kC_2 = -kC_1\cdot 0 + kC_2$ $\therefore$ C_1 and C_2 are arbitrary

$\lambda_m = (2m\pi)^2$; $y_m = \begin{cases}\cos(2m\pi x) \\ \text{OR} \\ \sin(2m\pi x)\end{cases}$

Math 314
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Take Home exam 1
March 3-17, 2009

Open book. No consultation with others.

(10) 1. Use series around x = 0 to solve x^2y" + xy' + (x^2 - 4)y = 0.

(10) 2. Solve y" + (x^2)y =0 in terms of Bessel functions by letting u = x^2/2, y = w (x^0.5).

(10) 3. Use Theorem 3 on page 194 and the table on A94 to evaluate: a) J_2(1.5); b) J_3(1.5).

(10) 4. Find the Fourier Series of f(x) = x on[-π, π].

(10) 5. Show -1 is an eigenvalue and find the corresponding eigenfunctions of:
y" +λ y = 0, (1 + e)y(0) + (e -1)y'(0) = 0 = (1 + e)y(1) + (1 - e)y'(1).

#2: $\frac{dy}{dx} = \frac{dw}{dx} x^{\frac{1}{2}} + \frac{1}{2} w x^{-1/2}$, $\frac{d^2y}{dx^2} = \frac{d^2w}{dx^2} x^{\frac{1}{2}} + \frac{dw}{dx} x^{-1/2} - \frac{1}{4} w x^{-3/2}$

$\therefore \frac{d^2w}{dx^2} x^{1/2} + \frac{dw}{dx} x^{-1/2} - \frac{1}{4} w x^{-3/2} + w x^{5/2} = 0$

$\frac{du}{dx} = x \therefore \frac{dw}{dx} = \frac{dw}{du} x$; $\frac{d^2w}{dx^2} = \frac{d^2w}{du^2} x^2 + \frac{dw}{du}$

$\therefore \left(\frac{d^2w}{du^2} x^2 + \frac{dw}{du}\right) x^{1/2} + \left(\frac{dw}{du} x\right) x^{-1/2} - \frac{1}{4} w x^{-3/2} + w x^{5/2} = 0$

$\therefore \frac{d^2w}{du^2} x^4 + 2x^2 \frac{dw}{du} + w\left(x^4 - \frac{1}{4}\right) = 0$

$\therefore \frac{d^2w}{du^2}(4u^2) + 2(2u)\frac{dw}{du} + w\left(4u^2 - \frac{1}{4}\right) = 0$

$\therefore \frac{d^2w}{du^2} u^2 + \frac{dw}{du} u + w\left(u^2 - \frac{1}{16}\right) = 0$

$\therefore w = c_1 J_{\frac{1}{4}}(u) + c_2 J_{-\frac{1}{4}}(u)$

$\therefore y = x^{1/2}\left[c_1 J_1(x^2/2) + c_2 J_2(x^2/2)\right]$

#3 a) $J_2(1.5) = \frac{2}{1.5} J_1(1.5) - J_0(1.5) = \frac{4}{3} \cdot 0.5579 - 0.5118 = .2321$

b) $J_3(1.5) = \frac{2 \cdot 2}{1.5} J_2(1.5) - J_1(1.5) = \frac{8}{3} \cdot .2321 - .5579 = .0610$

#4 $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0$; $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx\,dx = 0$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx = \frac{1}{n\pi}\left[-x\cos nx \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos nx\,dx\right] = \frac{-2}{n}(-1)^n$

So $\sum_{n=1}^{+\infty} \frac{2}{n}(-1)^{n+1} \sin nx$

#5 general solution of $y'' - y = 0$ is $y = c_1 e^{x} + c_2 e^{-x}$

To satisfy bdry conditions: $y' = c_1 e^{x} - c_2 e^{-x}$

$(1+e)(c_1+c_2) + (e-1)(c_1-c_2) = 0$ $\quad 2e\,c_1 + 2c_2 = 0$

$(1+e)(c_1 e + c_2 e^{-1}) + (1-e)(c_1 e - c_2 e^{-1}) = 0$ $\quad 2e\,c_1 + 2c_2 = 0$

So get nontrivial solution with $c_1 = 1$; $c_2 = -e$

$y = e^{x} - e^{-x+1}$

#1 $x^2y''+xy'+(x^2-4)y=0$; $a_m[(m+r)^2-4]=-a_{m-2}$; $r=\pm 2$

This is Bessel with $\nu=2$

With $r=-2$: $a_m m(m-4)=-a_{m-2}$; $a_0=a_1=a_2=a_3=0$; a_4 arbitrary

With $r=2$: $a_m(m+4)m=-a_{m-2}$; $a_m=0$ for m odd

$$a_{2m}=\frac{(-1)^m a_0}{2^{2m}m!\,3\cdot4\cdots(m+2)}$$ Let $y_1=\sum_{m=0}^{+\infty}a_{2m}x^{2m+2}$ with $a_0\equiv 1$

$$y_1=x^2\left[1-\frac{x^2}{2^2\cdot3}+\frac{x^4}{2^7\cdot3}-\frac{x^6}{2^9\cdot3^2\cdot5}+\cdots\right];\ y_2=uy_1$$

$$y_1^2=x^4\left[1-\frac{x^2}{2\cdot3}+\frac{7x^4}{2^6\cdot3^2}-\frac{x^6}{2^7\cdot3\cdot5}+\cdots\right]$$

$$u'=\frac{1}{y_1^2}e^{-\int\frac{dx}{x}}=\frac{1}{xy_1^2}=\left[x^{-5}+\frac{x^{-3}}{2\cdot3}+\frac{x^{-1}}{2^6}+\frac{19x}{2^7\cdot3^3\cdot5}+\cdots\right]$$

Thus $y_2=\frac{1}{2^6}y_1\ln x+x^{-2}\left(1-\frac{x^2}{2^2\cdot3}+\frac{x^4}{2^7\cdot3}-\frac{x^6}{2^9\cdot3^2\cdot5}\right)\left(\frac{-1}{2^2}-\frac{x^2}{2^2\cdot3}+\frac{19x^6}{2^8\cdot3\cdot5}+\cdots\right)$

$$=\frac{1}{2^6}y_1\ln x+x^{-2}\left(-\frac{1}{2^2}-\frac{x^2}{2^4}+\frac{29x^4}{2^9\cdot3^2}+\frac{19x^6}{2^{11}\cdot3^3}+\cdots\right)$$

Assume $y_2=y_1\ln x+\sum_{m=0}^{+\infty}b_mx^{m-2}$

$$2xy_1'+\sum_{m=0}^{+\infty}b_m[(m-2)(m-3)+(m-2)-4]x^{m-2}+\sum_{m=0}^{+\infty}b_mx^m=0$$

$$\sum_{m=0}^{+\infty}2(2m+2)a_{2m}x^{2m+2}+\sum_{m=0}^{+\infty}[m(m-4)b_m+b_{m-2}]x^{m-2}=0$$

$b_m\equiv 0$ for m odd, so after reindexing

$2(2m+2)a_{2m}+(2m+4)(2m)b_{2m+4}+b_{2m+2}=0$ for $m\ge -1$

$-4b_2+b_0=0$; $4a_0+b_2=0$; $2\cdot4a_2+2\cdot6b_6+b_4=0$

So if we take $a_0=1$, $b_4=29/(2^3\cdot3^2)$, we get

$$y_2=y_1\ln x+x^{-2}\left[-2^4-2^2x^2+\frac{29}{2^3\cdot3^2}x^4+\frac{19x^6}{2^5\cdot3^3}+\cdots\right]$$

Note that with this choice of b_4, this is 2^6 times first y_2

Note also $Y_2=\frac{2}{\pi}J_2\ln x-\frac{4}{\pi}x^{-2}-\frac{1}{\pi}+$ terms order x^2 or higher

If we multiply this by 4π, it agrees with answer above