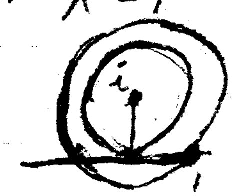
15.4 $\qquad$ 313 homework 12

#4 $\cos^2 z = \frac{1}{2}(1+\cos 2z) = \frac{1}{2}\left(1+\sum_{k=0}^{+\infty}\frac{(-1)^k(2z)^{2k}}{(2k)!}\right)$

$= 1 - z^2 + \frac{1}{3}z^4 - \frac{2}{45}z^6 \ldots \quad R = +\infty$

Could also use Cauchy product of $\cos x$ and $\cos x$

#6 $\frac{1}{z} = \frac{1}{1+(z-1)} = \sum_{k=0}^{+\infty}(-1)^k(z-1)^k = 1-(z-1)+(z-1)^2-(z-1)^3+\ldots \quad R=1$



#8 $\operatorname{Ln}(1-z) = \operatorname{Ln}[(1-i)-(z-i)] = \operatorname{Ln}\left[(1-i)\left(1-\frac{z-i}{1-i}\right)\right] =$

$\operatorname{Ln}(1-i) + \operatorname{Ln}\left(1-\left(\frac{z-i}{1-i}\right)\right) = \left(\ln\sqrt{2} - \frac{\pi}{4}i\right) - \sum_{k=0}^{+\infty}\left(\frac{z-i}{1-i}\right)^{k+1}/(k+1) =$

$\left(\ln\sqrt{2} - \frac{\pi}{4}i\right) - \left(\frac{1+i}{2}\right)(z-i) - \frac{i}{4}(z-i)^2 - \frac{(i-1)}{12}(z-i)^3 + \ldots$

Series converges for $\left|\frac{z-i}{1-i}\right| < 1$ i.e. $|z-i| < \sqrt{2}$, but only represents $\operatorname{Ln}(1-z)$ for $|z-i|<1$ since $\operatorname{Ln}(1-z)$ has sing at $z=0$

#14 $\int_0^z \frac{\sin t}{t}\,dt = \int_0^z \sum_{k=0}^{+\infty}\frac{(-1)^k t^{2k}}{(2k+1)!}\,dt = \sum_{k=0}^{+\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!\,(2k+1)} = z - \frac{z^3}{3!\cdot 3} + \frac{z^5}{5!\,5} + \ldots$

16.1

#2 $z\cos\frac{1}{z} = \sum_{k=0}^{+\infty}\frac{(-1)^k z^{1-2k}}{2k!} = z - \frac{z^{-1}}{2} + \frac{z^{-3}}{4!} - \frac{z^{-5}}{6!} + \ldots$ for $0<|z|<+\infty$

#6 $\frac{e^z}{z^2(1-z)} = \frac{1}{z^2}\sum_{n=0}^{+\infty}\frac{z^n}{n!}\sum_{n=0}^{+\infty}z^n = \sum_{n=0}^{+\infty}\sum_{k=0}^{n}\left(\frac{1}{k!}\right)z^{n-2} =$

$z^{-2} + 2z^{-1} + \frac{5}{2} + \frac{8}{3}z + \ldots$ for $0<|z|<1$

#10 $\frac{\cos z}{(z-\pi)^4} = \frac{\cos(\pi+z-\pi)}{(z-\pi)^4} = \frac{-\cos(z-\pi)}{(z-\pi)^4} = \sum_{k=0}^{+\infty}(-1)^{k+1}\frac{(z-\pi)^{2k-4}}{(2k)!} =$

$-(z-\pi)^{-4} + \frac{1}{2}(z-\pi)^{-2} - \frac{1}{4!} + \frac{1}{6!}(z-\pi)^2 + \ldots$ for $0<|z|<+\infty$

#16 $\frac{1}{1-z^2} = \frac{1}{(1-z)(1+z)} = \frac{1}{(1-z)}\,\frac{1}{(2+(z-1))} = \frac{1}{(1-z)2\left[1+\left(\frac{z-1}{2}\right)\right]} = \frac{1}{(1-z)2}\sum_{k=0}^{+\infty}(-1)^k\left(\frac{z-1}{2}\right)^k$

$= \frac{-1}{2}(z-1)^{-1} + \frac{1}{4} - \frac{1}{8}(z-1) + \frac{1}{16}(z-1)^2 + \ldots$ for $0<|z-1|<2$

$\frac{1}{1-z^2} = \frac{-1}{(z-1)^2(1+2/(z-1))} = \sum_{k=0}^{+\infty}(-1)^{k+1}\frac{2^k}{(z-1)^{2+k}} = \frac{-1}{(z-1)^2} + \frac{2}{(z-1)^3} - \frac{4}{(z-1)^4} + \ldots$

for $2<|z-1|$

#22 $\frac{1}{z^2} = \frac{1}{(i+(z-i))^2} = \frac{-1}{(1+(z-i)/i)^2} = -\sum_{n=0}^{+\infty}\binom{-2}{n}\left(\frac{z-i}{i}\right)^n = -1 + \frac{2}{i}(z-i) + 3(z-i)^2 + \ldots$

for $|z-i|<1$

$= \frac{1}{(z-i)^2\left[1+\left(\frac{i}{z-i}\right)\right]^2} = \sum_{n=0}^{+\infty}\binom{-2}{n}\frac{i^n}{(z-i)^{n+2}} = (z-i)^{-2} - 2i(z-i)^{-1} - 3 + \ldots$ for $1<|z-i|$

16.2

#2: at 0: pole of order 2; at ∞: $\frac{1}{w} + 2w - 3w^2$, simple pole

#6: $\cos z_0 - \sin z_0 = 0$ iff $\frac{1}{2}(e^{iz_0} + e^{-iz_0}) = \frac{1}{2i}(e^{iz_0} - e^{-iz_0})$ iff

$i(e^{2iz_0}+1) = e^{2iz_0} - 1$ iff $e^{2iz_0} = \frac{-1-i}{i-1} = i$ iff $2iz_0 = (\frac{\pi}{2} + 2\pi n)i$

iff $z_0 = \frac{\pi}{4} + n\pi$ Also $\cos z - \sin z = \cos(z_0 + (z-z_0)) - \sin(z_0 + (z - z_0))$

$= \cos z_0 \cos(z-z_0) - \sin(z_0)\sin(z-z_0) - \sin(z_0)\cos(z-z_0) - \cos z_0 \sin(z-z_0)$

$= -2\sin z_0 \sin(z-z_0) = -2\sin z_0[(z-z_0)\cdots]$ has simple zero

So the function has simple pole at these points

at ∞, singularity is not isolated

#14 $(z^4-16) = (z+2i)(z-2i)(z-2)(z+2)$ has simple zeros at $\pm 2, \pm 2i$

So $(z^4-16)^4$ has 4th order zeros at these points

#18 $(z^2-1)^2 = (z-1)^2(z+1)^2$ and $e^{z^2}-1 \neq 0$ at ± 1

so there are second order zeros at ± 1

$e^{z_0^2} = 1$ iff $z_0^2 = 2\pi n i$ iff $z_0 = \begin{cases} \pm\sqrt{2\pi n}\,\frac{(1+i)}{\sqrt{2}} & \text{for } n \geq 0 \\ \pm\sqrt{2\pi(-n)}\,\frac{(1-i)}{\sqrt{2}} & \text{for } n \leq 0 \end{cases}$

At these z_0; $\frac{d}{dz}(e^{z^2}-1)\big|_{z_0} = 2z_0 e^{z_0^2} \neq 0$ and $(z_0^2-1)^2 \neq 0$

So there are simple zeros at these z_0