313 homework

16.3 # 4; $\frac{\cos z}{z^6} = \sum_{n=0}^{+\infty} \frac{(-1)^n z^{2n-6}}{(2n)!}$ So $\underset{0}{Res} = a_{-1} = 0$

8 sing at z_0 iff $\cos(z_0) = \frac{1}{2}(e^{iz_0} + e^{-iz_0}) = 0$ iff $e^{2iz_0} = -1$

iff $2z_0 i = (\pi + 2n\pi)i$ iff $z_0 = (\frac{\pi}{2} + n\pi)$. For these z_0,

$\cos z = \cos(z_0 + z - z_0) = \cos z_0 \cos(z - z_0) - \sin(z_0)\sin(z - z_0)$

$= (-1)^{n+1} \sum_{k=0}^{+\infty} \frac{(-1)^k (z-z_0)^{2k+1}}{(2k+1)!} = (-1)^{n+1}[(z-z_0) + \cdots]$

So cos z has simple zero, so sec z has simple pole

OR $\frac{d}{dz}\cos z|_{z_0} = -\sin(z_0) = (-1)^{n+1}$. So $\cos z = 0 + (-1)^{n+1}(z-z_0) + \cdots$

So once again, cos z has simple zero at z_0

$(z - z_0)\sec z = \frac{(z-z_0)}{\cos z} \to \frac{1}{(-1)^{n+1}} = (-1)^{n+1} = \underset{z_0}{Res} \sec z$

12 ; $(z^4 - 1) = (z^2-1)(z^2+1) = (z-1)(z+1)(z-i)(z+i)$

For $z_0 = \pm 1$, $(z-z_0)f(z) = z^2/[(z+z_0)(z^2+1)] \to \frac{\pm 1}{4} = \underset{\pm 1}{Res}$

For $z_0 = \pm i$, $(z-z_0)f(z) = z^2/[(z^2-1)(z+z_0)] \to \frac{-1}{(-2)2z_0} = \frac{\mp i}{4} = \underset{\pm i}{Res}$

14 $\frac{\sin \pi z}{z^4} = \sum_{k=0}^{+\infty} \frac{(-1)^k \pi^{2k+1} z^{2k-3}}{(2k+1)!}$ So $\underset{0}{Res} = \frac{-\pi^3}{3!}$ Thus $\oint_c = \boxed{\frac{-\pi^4 i}{3}}$

18 As in problem 8, $\tan \pi z$ has simple pole at $z_0 = \frac{1}{2} + n$

The poles inside C are $\pm\frac{1}{2}, \pm\frac{3}{2}$. At these z_0, $(z-z_0)\tan \pi z =$

$\frac{(z-z_0)\sin \pi z}{(-1)^{n+1}(\pi z - \pi z_0)[1 + \cdots]} \to \frac{(-1)^n}{(-1)^{n+1}\pi} = \frac{-1}{\pi}$ Thus $\oint_c = 2\pi i \, 4(\frac{-1}{\pi}) = \boxed{-8i}$

22 3i is outside ; $\underset{0}{Res} = \frac{\cosh 0}{(0-3i)} = \frac{i}{3}$ Thus $\oint_c = 2\pi i(\frac{i}{3}) = \boxed{\frac{-2\pi}{3}}$

24 2 is outside ; for $z_0 = \frac{\pm i}{2}$, $\underset{z_0}{Res} = \frac{1 - 4z_0 + 6z_0^2}{2z_0(2-z_0)} =$

$\frac{-\frac{1}{2} - 4z_0}{4z_0 + \frac{1}{2}} = -1$ Thus $\oint_c = 2\pi i(-1 + -1) = \boxed{-4\pi i}$

16.4

#4 $\left[\frac{1}{8-2(z-\frac{1}{z})/(2i)}\right]\frac{1}{iz} = \frac{1}{8iz - z^2+1}$ has poles at $z_0 = \frac{-8i \pm \sqrt{-64+4}}{-2} = 4i \pm \sqrt{15}i$

$z_2 = i(4+\sqrt{15})$ is outside and $z_1 = i(4-\sqrt{15})$ is inside

$\operatorname{Res}_{z_1} = \frac{1}{-(z_1-z_2)} = \frac{-1}{-2\sqrt{15}i}$ ∴ $\int_0^{2\pi} d\theta = \oint_C = 2\pi i\left[\frac{1}{2\sqrt{15}i}\right] = \boxed{\frac{\pi}{\sqrt{15}}}$

#8 $\left[\frac{1+(z+\frac{1}{z})\frac{1}{2}}{17-8(z+\frac{1}{z})\frac{1}{2}}\right]\frac{1}{iz} = \frac{(2z^2+z+2)}{(-4z^2+17z-4)iz} = \frac{2z^2+z+2}{-4(z-\frac{1}{4})(z-4)zi}$

$\operatorname{Res}_0 = \frac{2}{-4i} = \frac{i}{2}$; $\operatorname{Res}_{\frac{1}{4}} = \frac{\frac{1}{8}+\frac{1}{4}+2}{-4(-15/4)\frac{1}{4}i} = \frac{-19i}{30}$ ∴ $\int_0^{2\pi} = 2\pi i\left(\frac{i}{2} - \frac{19i}{30}\right) = \boxed{\frac{4\pi}{15}}$

#10 $z_0 = \sqrt[4]{-1} = e^{\pi i(1+2\pi)/4} = (\pm 1+i)/\sqrt{2}, (\pm 1-i)/\sqrt{2}$; 1st 2 are in UHP

$\operatorname{Res}_{(1+i)/\sqrt{2}} = \frac{(1+i)/\sqrt{2}}{\sqrt{2}i(1+i)\sqrt{2}\sqrt{2}} = \frac{-i}{4}$; $\operatorname{Res}_{(-1+i)/\sqrt{2}} = \frac{i}{4}$ ∴ $\int_{-\infty}^{\infty} = \boxed{0}$ ∵ the integrand is odd

#16 $z_0 = \pm i, \pm 3i$ $\operatorname{Res}_i = \frac{1}{2i\cdot 8} = \frac{-i}{16}$; $\operatorname{Res}_{3i} = \frac{1}{(-8)6i} = \frac{i}{48}$

Thus $\int_{-\infty}^{+\infty} = 2\pi i\left[\frac{-i}{16}+\frac{i}{48}\right] = \boxed{\frac{\pi}{12}}$

#20 Same sing as in #10; $\operatorname{Res}_{(1+i)/\sqrt{2}} = \frac{e^{iz}}{\sqrt{2}i(1+i)\sqrt{2}\sqrt{2}} = \frac{e^{-1/\sqrt{2}}(\cos\frac{1}{\sqrt{2}}+i\sin\frac{1}{\sqrt{2}})}{2\sqrt{2}(1+i)i}$

$= -e^{-1/\sqrt{2}}(1+i)(\cos\frac{1}{\sqrt{2}}+i\sin\frac{1}{\sqrt{2}})/(4\sqrt{2})$

$\operatorname{Res}_{(-1+i)/\sqrt{2}} = e^{-1/\sqrt{2}}(1-i)(\cos\frac{1}{\sqrt{2}} - i\sin\frac{1}{\sqrt{2}})/(4\sqrt{2})$

Thus $\int_{-\infty}^{+\infty}\frac{\cos x\,dx}{x^4+1} = \operatorname{Re}\int_{-\infty}^{+\infty}\frac{e^{iz}dz}{z^2+1} = -2\pi\sum \operatorname{Im}(\operatorname{Res}) =$

$= -2\pi e^{-1/\sqrt{2}}\left[-\sin\frac{1}{\sqrt{2}} - \cos\frac{1}{\sqrt{2}} - \cos\frac{1}{\sqrt{2}} - \sin\frac{1}{\sqrt{2}}\right]/(4\sqrt{2}) = \pi e^{-1/\sqrt{2}}\left[\sin\frac{1}{\sqrt{2}}+\cos\frac{1}{\sqrt{2}}\right]/\sqrt{2}$

#24 $z_0 = \pm 1, \pm i$; $\operatorname{Res}_{\pm 1} = \frac{1}{\pm 4}$; $\operatorname{Res}_i = \frac{1}{4i}$

Thus $\int_{-\infty}^{+\infty} = 2\pi i\left[\frac{1}{4i}\right] + \pi i\left[\frac{1}{4} - \frac{1}{4}\right] = \boxed{\frac{\pi}{2}}$