Find the Maclaurin Series for $y = e^{x^2}\int_0^x e^{-t^2}\,dt$

Solution 1: $y = \sum_{j=0}^{+\infty} \frac{x^{2j}}{j!} \sum_{j=0}^{+\infty} \frac{(-1)^j x^{2j+1}}{j!(2j+1)} =$

$$\sum_{n=0}^{+\infty}\sum_{j=0}^{n} \frac{(-1)^j x^{2n+1}}{(n-j)!\,j!\,(2j+1)} = x + \frac{2}{3}x^3 + \frac{4}{15}x^5 + \cdots$$

$$\int_{-1}^{1}(1-x^2)^n\,dx = \int_{-1}^{1}\sum_{j=0}^{n}\binom{n}{j}(-1)^j(x)^{2j}\,dx = 2\sum_{j=0}^{n}\binom{n}{j}(-1)^j\frac{1}{2j+1}$$

Now let $u=(1-x)^k$, $u' = -k(1-x)^{k-1}$; $v' = (1+x)^{\ell}$; $v = \frac{1}{\ell+1}(1+x)^{\ell+1}$

Then $\int_{-1}^{1}(1-x)^k(1+x)^{\ell}\,dx = \left.\frac{(1-x)^k(1+x)^{\ell+1}}{\ell+1}\right]_{-1}^{1} + \int_{-1}^{1}\frac{k}{\ell+1}(1-x)^{k-1}(1+x)^{\ell+1}\,dx$

but first term on RHS is 0 for ℓ & k positive

Thus $\int_{-1}^{1}(1-x^2)^n\,dx = \int_{-1}^{1}(1-x)^n(1+x)^n\,dx = \frac{n}{n+1}\int_{-1}^{1}(1-x)^{n-1}(1+x)^{n+1}\,dx$

$$= \frac{n}{n+1}\,\frac{(n-1)}{n+2}\cdots\frac{1}{2n}\int_{-1}^{1}(1+x)^{2n}\,dx = \frac{n\cdot(n-1)\cdots 1}{(n+1)\cdots 2n}\,\frac{2^{2n+1}}{(2n+1)}$$

$$= \frac{(n!)^2\,2^{2n+1}}{(2n+1)!} = 2\sum_{j=0}^{n}\binom{n}{j}(-1)^j\frac{1}{2j+1}$$

Hence $$y = \sum_{n=0}^{+\infty}\frac{2^{2n}n!}{(2n+1)!}x^{2n+1} = \sum_{n=0}^{+\infty}\frac{2^n}{1\cdot 3\cdots(2n+1)}x^{2n+1}$$

Solution 2: $y' = 2xy + 1$ and $y(0) = 0$

$\sum_{k=0}^{+\infty} a_k x^k = y$; $a_0 = y(0) = 0$; $\sum_{k=0}^{+\infty} a_k k x^{k-1} = 2\sum_{k=0}^{+\infty} a_k x^{k+1} + 1$

$\sum_{k=-1}^{+\infty} a_{k+2}(k+2)x^{k+1} = 2\sum_{k=0}^{+\infty} a_k x^{k+1} + 1$; $a_1 = 1$

$a_{k+2} = \frac{2}{k+2}a_k$ for $k \geq 0$ $\therefore$ $a_k = 0$ for k even

$a_{2n+1} = \frac{2^n a_1}{(2n+1)(2n-1)\cdots 1} = \frac{2^{2n}n!}{(2n+1)!}$ $\therefore$ $y = \sum_{n=0}^{+\infty}\frac{2^{2n}n!}{(2n+1)!}x^{2n+1}$