Binomial Series

For pos. int. m, $(1+z)^m = \sum_{n=0}^{m} \binom{m}{n} z^m$ where $\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}$

Note $\binom{m}{n} = 0$ for $n > m$.

For neg. int. $m = -\ell$, $(1+z)^m = \frac{d^{(\ell-1)}}{dz^{(\ell-1)}}(1+z)^{-1}/(-1)(-2)\cdots(-\ell+1)$

$= \sum_{n=\ell-1}^{+\infty} (-1)^n n(n-1)\cdots(n-\ell+2) z^{n-\ell+1}/(-1)^{\ell-1}(\ell-1)!$ $\quad (k = n-\ell+1)$

$= \sum_{k=0}^{+\infty} \frac{(-1)^k (k+\ell-1)(k+\ell-2)\cdots(k+1)}{(\ell-1)!} z^k = \sum_{k=0}^{+\infty} \frac{(k+\ell-1)(k+\ell-2)\cdots\ell(-1)^k}{k!} z^k$

$= \sum_{k=0}^{+\infty} \frac{-\ell(-\ell-1)\cdots(-\ell-k+2)(-\ell-k+1)}{k!} z^k = \sum_{k=0}^{+\infty} \binom{-m}{k} z^k$ for $|z|<1$

Now consider $f(z) = (1+z)^m = e^{m\,\mathrm{Ln}(1+z)}$ for $m \in \mathbb{C}$; $|z|<1$

$f'(z) = m(1+z)^{m-1}$; $f^{(n)}(z) = m(m-1)\cdots(m-n+1)(1+z)^{m-n}$

$\therefore f^{(n)}(0)/n! = \frac{m(m-1)\cdots(m-n+1)}{n!} = \binom{m}{n}$

and $f(z) = \sum_{n=0}^{+\infty} \binom{m}{n} z^n$ for $|z|<1$

$\mathrm{Arcsin}\, z = -i\,\mathrm{Ln}(iz + \sqrt{1-z^2})$ $\therefore \frac{d}{dz}\mathrm{Arcsin}\, z = \frac{-i\left(i - \frac{2z}{2\sqrt{1-z^2}}\right)}{iz + \sqrt{1-z^2}}$

$= \frac{-i\,[i\sqrt{1-z^2} - z]}{\sqrt{1-z^2}\,[iz + \sqrt{1-z^2}]} = \frac{1}{\sqrt{1-z^2}}$ OR

$\sin(\mathrm{Arcsin}\, z) = z$ $\therefore \cos(\mathrm{Arcsin}\, z)\frac{d}{dz}\mathrm{Arcsin}\, z = 1$

but $\cos^2(\mathrm{Arcsin}\, z) + \sin^2(\mathrm{Arcsin}\, z) = 1$

$\therefore \cos(\mathrm{Arcsin}\, z) = \sqrt{1-z^2}$ so $\frac{d}{dz}\mathrm{Arcsin}\, z = \frac{1}{\sqrt{1-z^2}}$

$= \sum_{n=0}^{+\infty} \binom{-\frac{1}{2}}{n}(-1)^n z^{2n}$ $\therefore \mathrm{Arcsin}\, z = \sum_{n=0}^{+\infty} \binom{-\frac{1}{2}}{n}(-1)^n \frac{z^{2n+1}}{2n+1}$

$= z + \frac{1}{2}\frac{z^3}{3} + \frac{1}{2}\cdot\frac{3}{2}\frac{1}{2}\frac{z^5}{5} + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\frac{1}{2\cdot 3}\frac{z^7}{7}$