Summary of 10.6

Assume G is defined on S. Then $\iint_S G\,dS = \iint_R G|\vec{N}|\,dA$
where $\vec{r}: R \to S$ and $\vec{N} = \vec{r}_u \times \vec{r}_v$

Now assume $\vec{F}$ is defined on S and $\vec{n}$ are unit normals to S
Then $\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R \vec{F}\cdot\vec{n}\,|\vec{N}|\,dA = \pm\iint_R \vec{F}\cdot\vec{N}\,dA$
where $+$ if $\vec{n}$ and $\vec{N}$ are in same direction and $-$ if opposite.
Also $\vec{n} = [\cos\alpha, \cos\beta, \cos\gamma]$, so $\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)|\vec{N}|\,dA$
If $\vec{r}(x,y) = [x, y, f(x,y)]$, then $\vec{N} = [-f_x, -f_y, 1]$, so $\vec{N} = \vec{n}\sec\gamma$
and so $|\vec{N}| = |\sec\gamma|$ and $\iint_R F_3\cos\gamma\,|\vec{N}|\,dA = \pm\iint_R F_3\,dA$
where $+$ if $\cos\gamma > 0$ and $-$ if $\cos\gamma < 0$
Similarly $\iint_S F_2\cos\beta\,dS = \pm\iint_R F_2\,dA$ if $y = f(x,z)$
and $\iint_S F_1\cos\alpha\,dS = \pm\iint_R F_1\,dA$ if $x = f(y,z)$

Example 1: $\vec{F} = [3z^2, 6, 6xz]$; S is $y = x^2$, $0 \le x \le 2$, $0 \le z \le 3$
where $\vec{n}$ is $[-, +, 0]$

Let $\vec{r}(y,z) = [\sqrt{y}, y, z]$ for $0 \le y \le 4$ and $0 \le z \le 3$
then $-\int_0^3\int_0^4 3z^2\,dy\,dz = -\int_0^3 12z^2\,dz = -108$

Let $\vec{r}(x,z) = [x, x^2, z]$ for $0 \le x \le 2$ and $0 \le z \le 3$
then $\int_0^3\int_0^2 6\,dx\,dz = 36$ and $\iint_S (6xz)\,0\,dS = 0$

So $\iint_S \vec{F}\cdot\vec{n}\,dS = -108 + 36 = -72$. If $\vec{n}$ is in other direction $+72$

Example 2: $\vec{F} = [x^2, 0, 3y^2]$; S is $x+y+z = 1$, $0 \le x$, $0 \le y$, $0 \le z$
where $\vec{n}$ is $[+, +, +]$
Let $\vec{r}(x,y) = [x, y, 1-x-y]$ for $0 \le x \le 1-y$, $0 \le y \le 1$
Then $\int_0^1\int_0^{1-y} 3y^2\,dx\,dy = 3\int_0^1 (y^2 - y^3)\,dy = \frac{1}{4}$; $\iint_S 0\,dS = 0$

Let $\vec{r}(y,z) = [1-y-z, y, z]$ for $0 \le z \le 1-y$, $0 \le y \le 1$
Then $\int_0^1\int_0^{1-y} (1-y-z)^2\,dz\,dy = \int_0^1 \frac{1}{3}(1-y)^3\,dy = \frac{1}{12}$
So $\iint_S \vec{F}\cdot\vec{n}\,dS = \frac{1}{4} + 0 + \frac{1}{12} = \frac{1}{3}$