Math313 example

$\vec{r}(t) = b[\cos^3 t, \sin^3 t]$ for $0 \le t \le 2\pi$; $b > 0$

$\vec{r}'(t) = 3b[-\cos^2 t \sin t, \sin^2 t \cos t]$; $\frac{ds}{dt} = |\vec{r}'(t)| = 3b|\sin t \cos t|$

$\vec{u}(t) = \pm\langle -\cos t, \sin t\rangle$; ~~$\vec{r}''(t) = 3b[\quad]$~~

$\vec{r}''(t) = 3b[2\cos t \sin^2 t - \cos^3 t, 2\sin t \cos^2 t - \sin^3 t]$

$\vec{a}\cdot\vec{v} = 9b^2(\cos^5 t \sin t - \cos^3 t \sin^5 t) = 9b^2 \cos t \sin t (\cos^2 t - \sin^2 t)$

$\therefore \vec{a}_{tan} = \frac{9b^2(\cos^2 t - \sin^2 t)(\cos t \sin t)}{9b^2 \sin^2 t \cos^2 t} 3b[-\cos^2 t \sin t, \sin^2 t \cos t]$

$= 3b[-\cos t, \sin t](\cos^2 t - \sin^2 t)$

$\vec{a}_{norm} = \vec{a} - \vec{a}_{tan} = 3b \cos t \sin t [\sin t, \cos t]$

$\frac{d^2 s}{dt^2} = 3b(\cos^2 t - \sin^2 t)$ so $\vec{a}_{tan} = \frac{d^2 s}{dt^2}\vec{u} = 3b(\cos^2 t - \sin^2 t)[-\cos t, \sin t]$

$\frac{d\vec{u}}{dt} = [\sin t, \cos t]$ so $\vec{a}_{norm} = \frac{d\vec{u}}{dt}\frac{ds}{dt} = 3b \sin t \cos t [\sin t, \cos t]$

I assumed $\sin t \cos t \ge 0$. Note answers agree.

$K(s) = |\vec{u}'(s)| = |\vec{u}'(t)| / \frac{ds}{dt} = 1/3b|\cos t \sin t|$

$\vec{r}' \cdot \vec{r}' = 9b^2 \cos^2 t \sin^2 t$

$\vec{r}'' \cdot \vec{r}'' = 9b^2(\cos^6 t + \sin^6 t)$

$\vec{r}' \cdot \vec{r}'' = 9b^2(\cos^5 t \sin t - \sin^5 t \cos t) = 9b^2 \cos t \sin t (\cos^2 t - \sin^2 t)$

$\det(\vec{r}' \cdot \vec{r}')(\vec{r}'' \cdot \vec{r}'') - (\vec{r}' \cdot \vec{r}'') = 81b^4 \cos^2 t \sin^2 t (\cos^6 t + \sin^6 t - \cos^4 t + 2\cos^2 t \sin^2 t - \sin^4 t)$

$= 81b^4 \cos^4 t \sin^4 t$

$\therefore K(t) = \frac{9b^2 \cos^2 t \sin^2 t}{27b^3 |\cos^3 t \sin^3 t|} = \frac{1}{3b|\cos t \sin t|}$

Note answers agree again.