Math 313 Oct. 8-14, 2008
W. T. Kiley Take Home #1

You may use your books and notes. You may NOT communicate with anyone.
Clearly show your work. If you use software to get your answers, make it clear where you used it. Where possible, check your answers by doing the problem more than one way. Note that some surfaces and curves have more than one part.

(8) 1. For the curve in problem 26 of 9.5, find
a) unit tangent and unit normal;
b) velocity and speed;
c) the tangential and normal accelerations;
d) curvature.

(8) 2. problem 16 of 10.4

(8) 3. Problem 20 of 10.7

(8) 4. problem 6 of 10.9

#1 $\vec{r}(t) = [2\cos t, 2\sin t, 6t]$; $\vec{r}' = [-2\sin t, 2\cos t, 6]$

a) $\vec{u} = [-\sin t, \cos t, 3]/\sqrt{10}$; $\vec{n} = [\cos t, \sin t, 0]$

since $\vec{u}\cdot\vec{n} = 0$. Note there are many $\vec{n}$.

b) $\vec{v}(t) = \vec{r}\,'(t) = [-2\sin t, 2\cos t, 6]$; speed $= |\vec{v}(t)| = 2\sqrt{10} = \frac{ds}{dt}$

c) $\vec{a} = [-2\cos t, -2\sin t, 0]$; $\vec{a}\cdot\vec{v} = 0$ so $\boxed{\vec{a}_{tan} = 0}$; $\boxed{\vec{a}_{norm} = \vec{a}}$

OR $\vec{a}_{norm} = \frac{d\vec{u}}{dt}\frac{ds}{dt} = [-\cos t, -\sin t, 0]/\sqrt{10}\,(2\sqrt{10}) = 2[\cos t, \sin t, 0]$

$\vec{a}_{tan} = \vec{u}\,\frac{d^2 s}{dt^2} = 0$

d) $\kappa = |\vec{u}\,'(s)| = |\vec{u}\,'(t)\frac{dt}{ds}| = |[-\cos t, -\sin t, 0]/\sqrt{10}\,(1/2\sqrt{10})| = \boxed{\frac{1}{20}}$

OR $\vec{r}\,'\cdot\vec{r}\,' = 40$; $\vec{r}\,''\cdot\vec{r}\,'' = 4$; $\vec{r}\,'\cdot\vec{r}\,'' = 0$ so $\kappa = \sqrt{40\cdot 4 - 0}/(40)^{3/2} = \frac{2}{40} = \boxed{\frac{1}{20}}$

#2 $\nabla^2 w = 30x^4 y + 30xy^4$; $\iint_R (30x^4 y + 30xy^4)\, dA$

$= \int_0^{\pi}\int_0^2 r^6\, 30(\cos^4\theta\sin\theta + \cos\theta\sin^4\theta)\, dr\, d\theta$

$= 30\cdot 2^7/7\,[-\cos^5\theta/5 + \sin^5\theta/5]_0^{\pi} = \boxed{6\cdot 2^8/7}$ OR

$\nabla w = [6x^5 y + y^6, x^6 + 6xy^5]$; $\vec{r} = [2\cos\theta, 2\sin\theta]$; $0 \le \theta \le \pi$

$\vec{n} = [x/2, y/2]$ so $\int_C \frac{\partial w}{\partial n}\, ds = 2^7\int_0^{\pi}(6\cos^6\theta\sin\theta + \cos\theta\sin^6\theta + \sin\theta\cos^6\theta + 6\cos\theta\sin^6\theta)\, d\theta$

$= 2^7[-\cos^7\theta + \sin^7\theta]_0^{\pi} = 2^8$ for the top  over for bottom

$\vec{r} = [x, 0], -2 \le x \le 2$; $\vec{n} = [0,-1]$ so $\int_C \frac{\delta w}{\delta n} ds = \int_{-2}^{2} -x^6 dx = -\frac{2^8}{7}$

So $\oint_C \frac{\delta w}{\delta n} ds = 2^8 - 2^8/7 = \boxed{6\cdot 2^8/7}$

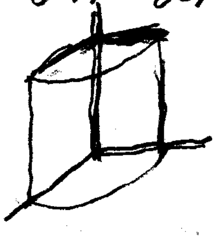
#3 $\vec{F} = [3xy^2, yx^2 - y^3, 3zx^2]$; $\nabla\cdot\vec{F} = 3y^2 + x^2 - 3y^2 + 3x^2 = 4x^2$

$\iiint_T 4x^2 dV = \iint_R \int_0^2 4x^2 dz\, dA = \int_0^{2\pi}\int_0^5 8r^3\cos^2\theta\, dr\, d\theta =$

$2\cdot 5^4 \int_0^{2\pi} \cos^2\theta\, d\theta = 2\cdot 5^4 \int_0^{2\pi} \frac{1}{2}(1+\cos 2\theta) d\theta = \boxed{2\cdot 5^4\pi}$ OR

on top $z=2, \vec{n} = [0,0,1]$, $\iint_S 6x^2 dS = \int_0^{2\pi}\int_0^5 6\cos^2\theta\, r^3\, dr\, d\theta$

$= \frac{3}{2}5^4\pi$



on bottom $z=0, \vec{n} = [0,0,1]$ so $\vec{F}\cdot\vec{n} = 0$

on side $\vec{r} = [5\cos u, 5\sin u, z]$ for $0 \le u \le 2\pi$; $0 \le z \le 2$

$\vec{N} = [5\cos u, 5\sin u, 0]$; $5^4\int_0^{2\pi}\int_0^2 [3\cos^2 u\sin^2 u + \cos^2 u \sin^2 u - \sin^4 u]\, dz\, du$

$= 2\cdot 5^4 \int_0^{2\pi} (5\cos^2 u \sin^2 u - \sin^2 u)\, du = 2\cdot 5^4 \int_0^{2\pi} (\frac{5}{4}(1-\cos^2 2u) - \frac{1}{2}(1-\cos 2u))\, du$

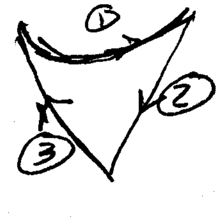
$= 2\cdot 5^4 (\frac{5}{4}(2\pi - \pi) - \frac{1}{2}2\pi) = 2\cdot 5^4 \cdot \frac{\pi}{4} = \frac{1}{2}5^4\cdot\pi$ so sum is $\boxed{2\cdot 5^4\cdot\pi}$

#4 $\nabla\times\vec{F} = [2y, 2z, 2x]$; $\vec{r} = [x, y, \sqrt{x^2+y^2}]$; R is $x^2+y^2 \le 4$; $0 \le y$

$\vec{N} = [-x/z, -y/z, 1]$; $\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS = \iint_R (-2xy/z - 2y + 2x)\, dA$

$= 2\int_0^{\pi}\int_0^2 (-r\cos\theta\sin\theta - r\sin\theta + r\cos\theta)\, r\, dr\, d\theta =$

$2\cdot\frac{8}{3}\int_0^{\pi} (-\cos\theta\sin\theta - \sin\theta + \cos\theta)\, d\theta = \boxed{-32/3}$ OR



on 1, $\vec{r} = [2\cos\theta, 2\sin\theta, 2]$; $\vec{r}' = [-2\sin\theta, 2\cos\theta, 0]$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi} [-8\sin\theta + 8\cos^3\theta)\, d\theta =$

$[8\cos\theta + 8\sin\theta - \frac{8}{3}\sin^3\theta]_0^{\pi} = -16$

on (2) $\vec{r} = [x, 0, -x]$ for $-2 \le x \le 0$ so $\int_{-2}^{0} x^2 dx = \frac{8}{3}$

on (3) $\vec{r} = [x, 0, x]$ for $0 \le x \le 2$ so $\int_0^2 x^2 dx = \frac{8}{3}$ $\quad -16 + \frac{8}{3} + \frac{8}{3} = \boxed{-\frac{32}{3}}$

Math 313 Exam #1
W. T. Kiley October 14, 2008

You may use one page of notes. Follow the Honor Code. Clearly show your work and answers.

(5) 1. Find the terminal point for the vector [2, -3, 5] with initial point (3, 4, -6).

$(3+2, 4-3, -6+5) = \boxed{(5, 1, -1)}$

(10) 2. a) Find the area of the parallelogram with sides [1, 1, 2] and [2, -1, 1].
b) Find the angle between the 2 vectors above.

a) $[1,1,2]\times[2,-1,1] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ 2 & -1 & 1 \end{vmatrix} = [1+2, 4-1, -1-2] = [3,3,-3]$

area $= |[3,3,-3]| = \boxed{3\sqrt{3}}$

b) $[1,1,2]\cdot[2,-1,1] = 3 = \sqrt{6}\sqrt{6}\cos\alpha$ so $\cos\alpha = \frac{1}{2}$, $\boxed{\alpha = \frac{\pi}{3}}$

Note $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\alpha$ gives $\sin\alpha = \frac{\sqrt{3}}{2}$ but α could be in second quad

(10) 3. a) Find the equation of a level surface that has parameterization
$r(u,v) = [2u \cos v, 2u \sin v, 5u]$, $0 \le v \le 2\pi$.
b) Find a parameterization for the surface x = 4y^2 + 9z^2.

a) $x^2+y^2 = 4u^2(\cos^2 v + \sin^2 v) = 4u^2 = \frac{4}{25}(5u)^2 = \frac{4}{25}z^2$

$\therefore \boxed{25x^2 + 25y^2 = 4z^2}$

b) $\vec{r}(y,z) = [4y^2+9z^2, y, z]$ OR

$\vec{r}(u,v) = [36u^2, 3u\cos v, 2u\sin v]$ for $0 \le v \le 2\pi$, $0 \le u$

(10) 4. Find normal vectors for the 2 surfaces above.

a) $[-50x, -50y, 8z]$ OR $\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2\cos v & 2\sin v & 5 \\ -2u\sin v & 2u\cos v & 0 \end{bmatrix} = [-10u\cos v, -10u\sin v, 4u]$

Note second is $\frac{1}{10}$ times first

b) $[1, -8y, -18z]$ OR $\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 72u & 3\cos v & 2\sin v \\ 0 & -3u\sin v & 2u\cos v \end{bmatrix} = [6u, -144u^2\cos v, -216u^2\sin v]$

Note second is $6u$ times first.

(10) 5. For the function f = x^2 + 3xy - 4z, find: a) the gradient; b) the directional derivative in the direction [2, 1, -2] at the point (5, -1, 3).

a) $\nabla f = \boxed{[2x+3y, 3x, -4]}$

b) $D_{[2,1,2]} = \nabla f(P)\cdot[2,1,-2]/3 = [7,15,-4]\cdot[2,1,-2]/3 = (14+15+8)/3 = \boxed{\frac{37}{3}}$

(10) 6. For the gradient found above, find: a) the divergence; b) the curl.

a) $\nabla\cdot\nabla f = \boxed{2}$

b) $\nabla\times\nabla f \equiv \boxed{0}$ always

(5) 7. Find a parameterization of the line from (2, 5, -3) to (6, 2, 1).

$\vec{b} = [6-2, 2-5, 1+3] = [4,-3,4]$; $\vec{r}(t) = [2+4t, 5-3t, -3+4t]$ for $0 \le t \le 1$

(10) 8. Find potentials, if they exist, for a) [2xy, sin z + x^2, y cos z]; b) [x^2, 2xy, 0].

a) $f_x = 2xy$ so $f = x^2y + g(y,z)$ ∴ $f_y = x^2 + g_y = \sin z + x^2$

∴ $g_y = \sin z$; $g = y\sin z + h(z)$; $y\cos z + h' = y\cos z$ ∴ $h' = 0$; h = const

$f = \boxed{x^2y + y\sin z}$

b) $f_x = x^2$ ∴ $f = x^3/3 + g(y,z)$ ∴ $f_y = g_y(y,z) = 2xy$ impossible

also $\nabla\times[x^2, 2xy, 0] \ne \vec{0}$ so poten D.N.E

(10) 9. Find line integrals $\int_C \vec{F}\,d\vec{r}$ if C is the curve in 7 and $\vec{F}$ is each of the functions in 8.

a) $\int_C \nabla f\cdot d\vec{r} = f(Q) - f(P) = \left[x^2y + y\sin z\right]_{(2,5,-3)}^{(6,2,1)} =$

$\boxed{72 + 2\sin 1 - (20 + 5\sin(-3))}$

b) $\vec{r}\,' = [4,-3,4]$ so $\int_0^1 [(2+4t)^2 4 + 2(2+4t)(5-3t)(-3)]\,dt =$

$\left[\frac{1}{3}(2+4t)^3 - 6(10t + 7t^2 - 4t^3\right]_0^1 = \frac{6^3}{3} - 6\cdot 13 - \frac{8}{3} = \boxed{\frac{-26}{3}}$